5/15/18 Lecture outline

⋆ Reading: Zwiebach chapters 8 and 9.

• Continue where we left off last time: symmetry and conservation laws for the case of Lorentz transformations. As we discussed, continuous symmetries on the worldsheet, δX^μ lead to conserved currents $j^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\delta\phi^a$, satisfying $\partial_\alpha j^\alpha = 0$, where $\alpha = 0, 1 \to \tau, \sigma$, and the corresponding worldsheet conserved charge is $\int d\sigma j^\tau$. For $\delta X^\mu = \epsilon^\mu$ translations, the charge is $p_\mu = \int d\sigma \mathcal{P}^\tau_\mu$. Now continue with Lorentz symmetry, which comes from the worldsheet symmetry $\delta X^\mu = \epsilon^{\mu\nu} X_\nu$, which is a symmetry if $\epsilon^{\mu\nu} = \epsilon^{[\mu\nu]}$, e.g. $\delta(\eta_{\mu\nu} X^\mu X^\nu) = 0$. Discuss cases of spatial rotations and boosts, explain why both indeed involve antisymmetric $\epsilon^{\mu\nu}$. Lorentz symmetry is of course a symmetry of the string Lagrangian $\mathcal{L}_{NG} = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}$ involve $\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$, since all Lorentz vector indices are contracted via Lorentz scalar dot products.

The associated conserved currents are $\mathcal{M}^\alpha_{\mu\nu} = X_\mu \mathcal{P}^\alpha_\nu - (\mu \leftrightarrow \nu)$. The corresponding charges $M_{\mu\nu} = \int(\mathcal{M}^\tau_{\mu\nu} d\sigma$ are the angular momentum. We can also consider more generally conserved charges $M_{\mu\nu}[\Gamma] = \int_\Gamma (\mathcal{M}^\tau_{\mu\nu} d\sigma - \mathcal{M}^\sigma_{\mu\nu} d\tau)$. Note that the charges associated with boosts are $M^{0i} = ctp^i - \int d\sigma X^i \mathcal{P}^{\tau 0}$, which can be interpreted as $X^i_{cm}(t) = \frac{-cM^{0i}}{E} + t\frac{c^2 p^i}{E}$.

• Recall $J = \hbar\alpha' E^2$, with $[\alpha'] = -2$, which is the Regge trajectory observation of the early '70s.. Consider now a string rotating in 12 plane, with the EOM solved by (as discussed last week): $\vec{X} = \frac{\sigma_1}{\pi}\cos(\pi\sigma/\sigma_1)(\cos(\pi ct/\sigma_1), \sin(\pi ct/\sigma_1))$. So $\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2}\dot{\vec{X}} = \frac{T_0}{c}\cos(\pi\sigma/\sigma_1)(-\sin(\pi ct/\sigma_1), \cos(\pi ct/\sigma_1))$. Find that the rotational angular momentum has $M_{12} = \int_0^{\sigma_1} d\sigma (X_1 \mathcal{P}^\tau_2 - X_2 \mathcal{P}^\tau_1)$, which using above $\vec{X}(t,\sigma)$ and $\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2}\partial_t \vec{X}$, leads to $M_{12} = \sigma_1^2 T_0/2\pi c$, which is a constant as expected. Since $\sigma_1 = E/T_0$ and $M_{12} = J$, this gives $J = \alpha'\hbar E^2$, with $T_0 \equiv 1/2\pi\alpha'\hbar c$. The string length is $\ell_s = \hbar c\sqrt{\alpha'}$.

• Aside, for later: the string worldsheet analog of $S_{particle} \supset \int qA_\mu dx^\mu$ is $S_{string} \supset -\int_\Sigma B_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu d\sigma d\tau$.

• As some review for the next topic, recall the light cone coordinates: $a^\pm = (a^0 \pm a^1)/\sqrt{2}$, so $a \cdot b = -a^- a^+ - a^+ a^- + \sum_I a^I b^I$, where $I = 2, \dots$ runs over the transverse space directions. Ugly, but can help quantize. Example from QM: in non-relativistic case, get Schrodinger equation by writing $H = \vec{p}^2/2m + V(x)$ and replacing $H \to i\hbar\frac{\partial}{\partial t}$ and $\vec{p} \to -i\hbar\nabla$. In relativistic case, considering free particle for simplicity, have $H = \sqrt{(c\vec{p})^2 + (mc^2)^2}$ and $\vec{p} \to -i\hbar\nabla$ would require understanding how to take the square-root of an operator. This is what led Dirac to the Dirac equation for relativistic electrons, and the start of quantum field theory. Very interesting and long story, but not the topic of this class. We will avoid going there by the trick of the light cone.

Write $-p\cdot p = m^2$ (setting $c=1$) as $2p^+p^- = \sum_I p^I p^I + m^2$. In the light cone, we think of x^+ as time. Then $p^- = H_{lc} = (\sum_I p^I p^I + m^2)/2p^+$. No need for square-root. Looks similar to non-relativistic case.

• Use $\hbar = c = 1$ units. Recall $[\alpha'] = 1/[T_0] = L^2$. Write

$$\mathcal{L}_{NG} = -\frac{1}{2\pi\alpha'}\sqrt{(\dot X\cdot X')^2 - (\dot X)^2(X')^2},$$

and we have

$$\mathcal{P}^\tau_\mu = \frac{\partial\mathcal{L}}{\partial \dot X^\mu} = -\frac{1}{2\pi\alpha'}\frac{(\dot X\cdot X')X'_\mu - (X')^2\dot X_\mu}{\sqrt{(\dot X\cdot X')^2 - (\dot X)^2(X')^2}},$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial\mathcal{L}}{\partial X^{\mu\prime}} = -\frac{1}{2\pi\alpha'}\frac{(\dot X\cdot X')\dot X_\mu - (\dot X)^2 X'_\mu}{\sqrt{(\dot X\cdot X')^2 - (\dot X)^2(X')^2}}.$$

which we simplified by picking static gauge.

• Generalize static gauge (to eventually get to light cone gauge). Consider e.g. gauge $n_\mu X^\mu = \lambda\tau$ for time-like n_μ. Static gauge is $n_\mu = (1,0,\ldots,0)$. Vary, $n_\mu dX^\mu = \lambda d\tau$, so n_μ is orthogonal to the string tangent at constant τ. We want dX^μ along the string to be spacelike (or null at isolated points, e.g. the Neumann open string endpoints).