4/17/18 Lecture outline

⋆ Reading: Zwiebach chapter 3.

• The action for a relativistic point particle of mass m is $S = -mc\int ds = -mc^2\int dt\sqrt{1-v^2/c^2}$. This gives $\vec{p} = \partial_{\vec{v}} = \gamma m\vec{v}$ and $H = \vec{p}\cdot\vec{v} - L = \gamma mc^2$, both of which are constants of the motion (thanks to the time and spatial translation invariance).

When the particle is charged and in the presence of electric and magnetic fields, there is the new term in the action

$$S = \int(-mcds + \frac{q}{c}A_\mu dx^\mu), \tag{1}$$

which is manifestly relativistically invariant (and also reparameterization) invariant. Note also that, under a gauge transformation, we have $S \to S + \frac{qf}{c}|_{endpoints}$, which does not affect the equations of motion (just as changing the Lagrangian by a total time derivative does not).

The lagrangian is thus $L = -mc\sqrt{1-\vec{v}^2/c^2} + \frac{q}{c}\vec{v}\cdot\vec{A} - q\phi$. The momentum conjugate to $\vec{r}$ is $\vec{P} = \partial L/\partial\vec{v} = m\vec{v}/\sqrt{1-\vec{v}^2/c^2} + \frac{q}{c}\vec{A}$. The Hamiltonian is $H = \vec{v}\cdot\vec{P} - L = \sqrt{m^2c^4 + c^2(\vec{P} - \frac{q}{c}\vec{A})^2} + q\phi$. The equations of motion can be written as $\frac{d^2x^\mu}{d\tau^2} = \frac{q}{mc}F_{\mu\nu}\frac{dx^\nu}{d\tau}$. In the non-relativistic limit we have $H = \frac{1}{2m}(\vec{P} - \frac{q}{c}\vec{A})^2 + q\phi$, where $\vec{P} - \frac{q}{c}\vec{A} = m\vec{v}$.

• In QM, gauge transformation $A^\mu \to A^\mu + \partial^\mu f$ accompanies giving an overall, local phase to the QM wavefunction $\psi \to e^{-iqf/\hbar c}\psi$, where q is the electric charge of the field.

Can form covariant derivatives $D_\mu\psi = (\partial_\mu + i(q/\hbar c)A_\mu)\psi$ so $D_\mu\psi \to e^{-iqf/\hbar c}D_\mu\psi$ under a gauge transformation.

• Maxwell theory and gravity in general D spacetime dimensions. $ds^2_{flat} = -c^2dt^2 + dx_1^2 + \ldots dx^2_{D-1}$. For any D, we have the same Maxwell's equations, so $F^{\mu\nu} = \partial^{[\mu}A^{\nu]}$ and $\partial_\mu F^{\mu\nu} = \frac{1}{c}j^\nu$. A point charge q has $\rho = q\delta^{D-1}(\vec{x})$ and makes an electric field with $\nabla\cdot\vec{E} = q\delta^d(\vec{x})$ in a world with $D = d+1$ spacetime dimension (the +1 is the time dimension, and there are d spatial directions), so $\int_{S^{d-1}}\vec{E}\cdot d\vec{a} = q$. Thus $\vec{E} = E(r)\hat{r}$ with $E(r) = q/r^{d-1}vol(S^{d-1})$, where $vol(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of a unit sphere[1] surrounding the charge. Finally, we get that a point charge makes electric field given by $E(r) = \Gamma(d/2)q/2\pi^{d/2}r^{d-1}$. For $d=3$, get $E(r) = q/4\pi r^2$, good.

• What about gravity in other D? In 4d, we have gravitational potential given by $V_g^{(4)} = -GM/r$, which solves $\nabla^2 V_g^{(D)} = 4\pi G^{(D)}\rho_m$. This is the gravitational potential

[1] To show this, use $\int\prod_{i=1}^d dx_i e^{-x_i^2} = \pi^{d/2} = \int drr^{d-1}d\Omega_{d-1}e^{-r^2} = \Omega_{d-1}\frac{1}{2}\int_0^\infty dtt^{d/2-1}e^{-t} = \frac{1}{2}\Omega_{d-1}\Gamma(d/2)$.

equation in any spacetime dimension, with gravitational force taken to be $F = -m\nabla V_g$. In $\hbar = c = 1$ units, get $G = \ell_P^{D-2}$ in D spacetime dimensions. Get $G^D = GV_C$, where V_C is the compactification volume.

- The electric and magnetic fields themselves have a lagrangian, with action

$$S = \int d^D x \mathcal{L}, \qquad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{c} A_\mu j^\mu.$$

The two Maxwell's equations expressing absence of magnetic monopoles are, again, solved by setting $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$. The other two Maxwell's equations then come from the Euler -Lagrange equations of the above action upon varying $A_\mu \to A_\mu + \delta A_\mu$: the action is stationary when

$$\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0.$$

It is convenient to rescale A_μ and j^μ such that the unit of electric charge is 1 instead of the charge e of an electron. Also, I will use g instead of e. Doing so, g only appears in the kinetic terms for the gauge fields: $\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu}$. This applies in any D, and in any D, the mass dimensions are $[F_{\mu\nu}] = 2$ and $[\mathcal{L}] = D$, so $[g^{-2}] = D - 4$. The force between two point charges separated by distance r is $\sim g^2 r^{1-d}$ and $[F] = [ma] = 2 = 4 - D + d - 1$ checks.

If we dimensionally reduce, then $g^{-2}_{reduced} = g^{-2}_{original} V$.