5/23/19 Lecture outline

⋆ Reading: Zwiebach chapters 10, 11

• Recall from last time: Maxwell field A^μ and quantize → photons. In the vacuum, setting $j^\mu = 0$, we have $\partial_\mu F^{\nu\mu} = 0$, which implies $\partial^2 A^\mu - \partial^\mu(\partial \cdot A) = 0$. Massless. Fourier transform to $A^\mu(p)$, with $A^\mu(-p) = A^\mu(p)^*$, and get $(p^2\eta^{\mu\nu} - p^\mu p^\nu)A_\nu(p) = 0$. Gauge invariance: $\delta A_\mu(p) = ip_\mu \epsilon(p)$. In light cone gauge, since $p^+ \neq 0$, can use gauge invariance to choose ϵ such that $A^+(p) = 0$. Then taking $\mu = +$ in the EOM, and $p^+ \neq 0$, get $\partial \cdot A = 0$ which gives $A^- = (p^I A^I)/p^+$, i.e. A^- is not an independent d.o.f., but rather constrained, and the Maxewell EOM gives $p^2 A^\mu(p) = 0$. For $p^2 \neq 0$, require $A^\mu(p) = 0$, and for $p^2 = 0$ get that there are $D-2$ physical transverse d.o.f., the $A^I(p)$. The one-photon states are

$$\sum_{I=2}^{D-1} \xi_I a^{I\dagger}_{p^+,p_T}|\omega\rangle.$$

• Likewise, in GR in the weak field expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is treated as a small perturbation, Einstein's equations of GR in a linearized expansion shows that $h_{\mu\nu}$ has wave solutions propagating at $v = c$. The linearized EOM in terms of the Fourier transform $h_{\mu\nu}(p)$ is $p^2 h^{\mu\nu} - p_\alpha(p^\mu h^{\nu\alpha} + p^\nu h^{\mu\alpha}) + p^\mu p^\nu h = 0$. The gauge transformation (general coordinate invariance) is $\delta h^{\mu\nu} = ip^\mu \epsilon^\nu(p) + (\mu \leftrightarrow \nu)$. The EOM are indeed invariant under that. In light cone coordinates, $\delta h^{++} = 2ip^+\epsilon^+$, $\delta h^{+I} = ip^+\epsilon^- + ip^-\epsilon^+$, and $\delta h^{+I} = ip_+\epsilon^I + ip^I\epsilon^+$. So can chose $\epsilon^\pm$ and ϵ^I to impose the gravitational light cone gauge conditions: $h^{++} = h^{+-} = h^{+I} = 0$. The $h^{-\mu}$ Other components are constrained. The equations of motion, with $p_+ \neq 0$, imply that $h^{IJ}\delta_{IJ} = 0$. So physical d.o.f. are specified by a traceless symmetric matrix h^{IJ} in the $D-2$ transverse directions. So there are $\frac{1}{2}D(D-3)$ d.o.f., e.g. in 4d gravity waves have two independent polarizations (they are sometimes called $+$ and x polarizations, or linear combinations are called L and R circular).

• Recall the relativistic point particle, with $S = \int L d\tau$ and $L = -m\sqrt{-\dot{x}^2}$, where $\dot{\equiv}\frac{d}{d\tau}$. ($\tau$ is taken to be dimensionless.) The momentum is $p_\mu = \partial L/\partial \dot{x}^\mu = m\dot{x}_\mu/\sqrt{-\dot{x}^2}$ and the EOM is $\dot{p}_\mu = 0$. In light cone gauge we take $x^+ = p^+\tau/m^2$. Then $p^+ = m\dot{x}^+/\sqrt{-\dot{x}^2}$ and the light cone gauge condition implies $\dot{x}^2 = -1/m^2$, so $p_\mu = m^2\dot{x}_\mu$. Also, $p^2 + m^2 = 0$ yields $p^- = (p^I p^I + m^2)/2p^+$, which is solved for p^- and then $\dot{x}^- = p^-/m^2$ is integrated to $x^- = p^-\tau/m^2 + x_0^-$. Also, $x^I = x_0^I + p^I\tau/m^2$. The independent variables are (x^I, x_0^-, p^I, p^+).

• Heisenberg picture: put time dependence in the operators rather than the states, with $[q(t), p(t)] = i$ and

$$i\frac{d}{dt}\mathcal{O}(t) = i\frac{\partial \mathcal{O}}{\partial t} + [\mathcal{O}, H].$$

For time independent Hamiltonian, we have $|\psi(t)\rangle_S = e^{-iHt}|\psi\rangle_H$ and $\mathcal{O}_H = e^{iHt}\mathcal{O}_S e^{-iHt}$.

• Quantize the point particle in light cone gauge by taking the independent operators (x^I, x_0^-, p^I, p^+), with $[x^I, p^J] = i\eta^{IJ}$ and $[x_0^-, p^+] = i\eta^{-+} = -i$. These commutators are for either S or H picture, with the operators being functions of τ in the H picture. The remaining variables are defined by $x^+(\tau) = p^+\tau/m^2$, $x^-(\tau) = x_0^- + p^-\tau/m^2$, $p^- = (p^I p^I + m^2)/2p^+$ (the first two are explicitly τ dependent even in the S picture).

The Hamiltonian is $\sim p^-$, which generates $\frac{\partial}{\partial x^+}$ translations. Since $\frac{\partial}{\partial \tau} = \frac{p^+}{m^2}\frac{\partial}{\partial x^+} \leftrightarrow \frac{p^+}{m^2}p^-$ the Hamiltonian is

$$H = \frac{p^+ p^-}{m^2} = \frac{1}{2m^2}(p^I p^I + m^2).$$

Verify e.g.

$$i\frac{d}{d\tau}p^\mu = [p^\mu, H] = 0, \qquad i\frac{dx^I}{d\tau} = [x^I, H] = i\frac{p^I}{m^2},$$

reproducing the correct EOM. Likewise, verify $\dot{x}_0^- = 0$ and $\dot{x}^+ = \partial_\tau x^+ = p^+/m^2$. The momentum eigenstates are labeled by $|p^+, p^I\rangle$ and these are also energy eigenstates, $H|p^+, p^I\rangle = \frac{1}{2m^2}(p^I p^I + m^2)|p^+, p^I\rangle$.

• Connect the quantized point particle with the excitations of scalar field theory via

$$|p^+, p^I\rangle \leftrightarrow a^\dagger_{p^+, p^I}|\Omega\rangle.$$

The S.E. of the quantum point particle wavefunction maps to the classical scalar field equations, e.g. in light cone gauge:

$$(i\partial_\tau - \frac{1}{2m^2}(p^I p^I + m^2))\phi(\tau, p^+, p^I) = 0$$

is either the quantum S.E. of the point particle or the classical field equations of a scalar field.

(Aside: the light cone is here used as a trick to get to "second quantization." "First quantization" is what you learn the first time you study (non-relativisitic) QM: replace coordinates and momenta with operators, and Poisson brackets with commutators. Second quantization is for field theory, replacing the fields and their conjugate momenta with operators, and their PBs with commutators, leading to multi-particle states. Here light-cone first quantization of the point particle leads to a Schrodinger equation that agrees with the classical EOM of a light-cone field theory, which we then need to quantize again to get second quantization.