4/16/19 Lecture outline

⋆ Reading: Zwiebach chapters 2 and 3.

• Last time: $f^{\mu}_{E\&M} = \frac{q}{c}F^{\mu\nu}u_\nu$, $F^{\mu\nu} = \partial^{[\mu}A^{\nu]}$, $\partial_\lambda F^{\mu\lambda} = \frac{1}{c}j^\mu$. Light waves have $A^\mu = \epsilon_\mu e^{ik\cdot x} + h.c.$ with $k^2 = 0$ and $D-2$ independent polarizations.

• The action for a relativistic point particle of mass m is $S = -mc\int ds = -mc^2\int dt\sqrt{1-v^2/c^2}$. This gives $\vec{p} = \partial_{\vec{v}} = \gamma m\vec{v}$ and $H = \vec{p}\cdot\vec{v} - L = \gamma mc^2$, both of which are constants of the motion (thanks to the time and spatial translation invariance).

When the particle is charged and in the presence of electric and magnetic fields, there is the new term in the action

$$S = \int(-mcds + \frac{q}{c}A_\mu dx^\mu), \tag{1}$$

which is manifestly relativistically invariant (and also reparameterization) invariant. Note also that, under a gauge transformation, we have $S \to S + \frac{qf}{c}$, which does not affect the equations of motion (just as changing the Lagrangian by a total time derivative does not).

The lagrangian is thus $L = -mc\sqrt{1-\vec{v}^2/c^2} + \frac{q}{c}\vec{v}\cdot\vec{A} - q\phi$. The momentum conjugate to $\vec{r}$ is $\vec{P} = \partial L/\partial\vec{v} = m\vec{v}/\sqrt{1-\vec{v}^2/c^2} + \frac{q}{c}\vec{A}$. The Hamiltonian is $H = \vec{v}\cdot\vec{P} - L = \sqrt{m^2c^4 + c^2(\vec{P} - \frac{q}{c}\vec{A})^2} + q\phi$. The equations of motion can be written as $\frac{d^2x^\mu}{d\tau^2} = \frac{q}{mc}F_{\mu\nu}\frac{dx^\nu}{d\tau}$. In the non-relativistic limit we have $H = \frac{1}{2m}(\vec{P} - \frac{q}{c}\vec{A})^2 + q\phi$, where $\vec{P} - \frac{q}{c}\vec{A} = m\vec{v}$.

• In QM, gauge transformation $A^\mu \to A^\mu + \partial^\mu f$ accompanies giving an overall, local phase to the QM wavefunction $\psi \to e^{iqf/\hbar c}\psi$, where q is the electric charge of the field.

Can form covariant derivatives $D_\mu\psi = (\partial_\mu - i(q/\hbar c)A_\mu)\psi$ so $D_\mu\psi \to e^{-iqf/\hbar c}D_\mu\psi$ under a gauge transformation.

• Maxwell theory and gravity in general D spacetime dimensions. $ds^2_{flat} = -c^2dt^2 + dx_1^2 + \dots dx^2_{D-1}$. For any D, we have the same Maxwell's equations, so $F^{\mu\nu} = \partial^{[\mu}A^{\nu]}$ and $\partial_\mu F^{\mu\nu} = \frac{1}{c}j^\nu$. A point charge q has $\rho = q\delta^{D-1}(\vec{x})$ and makes an electric field with $\nabla\cdot\vec{E} = q\delta^d(\vec{x})$ in a world with $D = d+1$ spacetime dimension (the $+1$ is the time dimension, and there are d spatial directions), so $\int_{S^{d-1}}\vec{E}\cdot d\vec{a} = q$. Thus $\vec{E} = E(r)\hat{r}$ with $E(r) = q/r^{d-1}vol(S^{d-1})$, where $vol(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of a unit sphere[1] surrounding the charge. Finally, we get that a point charge makes electric field given by $E(r) = \Gamma(d/2)q/2\pi^{d/2}r^{d-1}$. For $d = 3$, get $E(r) = q/4\pi r^2$, good.

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[1] To show this, use $\int\prod_{i=1}^d dx_i e^{-x_i^2} = \pi^{d/2} = \int dr r^{d-1}d\Omega_{d-1}e^{-r^2} = \Omega_{d-1}\frac{1}{2}\int_0^\infty dt t^{d/2-1}e^{-t} = \frac{1}{2}\Omega_{d-1}\Gamma(d/2)$.

• What about gravity in other D? In 4d, we have gravitational potential given by $V_g^{(4)} = -GM/r$, which solves $\nabla^2 V_g^{(D)} = 4\pi G^{(D)} \rho_m$.

In any spacetime dimension, take $F_g = -m \nabla \Phi_g$, and $\nabla^2 \Phi_g^{(D)} = 4\pi G^{(D)} \rho_m$. So in $\hbar = c = 1$ units, get $[F] = 2$, $[\Phi] = 0$, $[\rho] = D$, so $[G] = 2 - D$. Take $G = \ell_P^{D-2}$ in D spacetime dimensions.