215c Homework exercises 6, Spring 2020, due May 18

1. Consider a gauge theory with general gauge group $G$ and a Dirac Fermion in a general representation $r$. Consider the process $\psi + g \to \psi + g$ at tree-level, where $g$ denotes the gauge field. Write down the tree-level amplitude and verify that it vanishes if the polarization of either external gluon is longitudinal, i.e. equal to its momentum. Hint: there is an extra diagram as compared with QED, which is important because the order of the group generators in other diagrams matters for non-Abelian groups.

2. The group $SU(5)$ contains $SU(3) \times SU(2) \times U(1)$ as a subgroup. The fundamental representation of $SU(5)$ is the $\mathbf{5}$, which we can write as $\psi^{c=1\ldots5}$, and the anti-fundamental is the $\bar{5}$ which we can write as $\tilde{\psi}_{c=1\ldots5}$. Some multiplication rules are $\mathbf{5} \times \mathbf{5} = \mathbf{10}_A + \mathbf{15}_S$, where the subscripts are for the anti-symmetric and symmetric products, $\psi_{10}^{cd} = -\psi_{10}^{dc}$ and $\psi_{15}^{cd} = \psi_{15}^{dc}$. Also, $\mathbf{5} \times \bar{\mathbf{5}} = \mathbf{1} + \mathbf{24}$, where the $\mathbf{1}$ is $\delta^c_d$ and $(\psi_{24})^c_d$ is traceless, $(\psi_{24})^c_c = 0$, and it is the adjoint representation.

   (a) Under $SU(5) \to SU(3) \times SU(2) \times U(1)$, the fundamental decomposes as $\mathbf{5} \to (\mathbf{3}, 1)_{q_3} + (1, \mathbf{2})_{q_2}$, where $q_3$ and $q_2$ are the $U(1)$ charge. Write the $U(1)$ generator in terms of the $(\psi_{24})^c_c = 0$ and, using also the normalization condition that $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$ for the fundamental, determine $q_3$ and $q_2$.

   (b) Write the similar decompositions of $\bar{\mathbf{5}}$, $\mathbf{10}$, and $\mathbf{24}$ under $SU(5) \to SU(3) \times SU(2) \times U(1)$. Recall that $2 \cong \bar{2}$ for $SU(2)$.

   (c) The fermions in a generation of the SM are in the $SU(3) \times SU(2) \times U(1)_Y$ representations $(3, 2)_{1/3} + (\bar{3}, 1)_{-4/3} + (\bar{3}, 1)_{2/3} + (1, 2)_{-1} + (1, 1)_2$. Compare with the results from the previous parts and thus show that they fit into $SU(5)$ representations.

   (d) Using the results from the previous parts, write down the $SU(3) \times SU(2) \times U(1)_Y$ representations of the missing $SU(5)$ gauge fields, that are not in $SU(3) \times SU(2) \times U(1)_Y$.

3. Consider a massless, Dirac Fermion $\psi$ in a general representation $r$ of a general non-Abelian gauge group $G$, with $\mathcal{L} = \bar{\psi} i \not{D} \psi - \frac{1}{2}\mathrm{Tr} F_{\mu\nu} F^{\mu\nu}$. Use the Feynman rules to write down the one-loop correction to the Fermion's propagator. Note that the result is similar to the one-loop correction to the electron's propagator in QED, up to a difference in the group theory factors that you should determine. Using this, relate the anomalous dimension of the Fermion in representation $r$ to the anomalous dimension of the electron in QED (which you might have computed last quarter).