215c, 3/30/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ There will sometimes be significant overlap between my lectures and **David Tong's beautiful lecture notes on gauge theories**. When I think about how I would like to present things, the order of topics, the notation, opportunities to spice things up by sprinkling some modern tidbits on classic fundamentals, etc – I then find that David has already beautifully presented it, almost exactly as I would do it. Thank you, David! http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

⋆ **Week 1 reading: Tong chapter 1, and start chapter 2.**

• Symmetries are important, and appear in several varieties, in this topic and class. First, there is the rotational symmetry of space, which is classically $SO(3)$ but actually we want its double cover, $SU(2)$, to allow for Fermions. I will only consider relativistic QFTs in this class (of course, non-relativistic QFTs are also interesting), so $SO(3)$ is extended to $SO(1,3)$, or the version with spinors, $spin(1,3)$, or $SU(2)_L \times SU(2)_R$ in Euclidean space. Associated with this, we can write left and right handed chiral Fermion fields ψ_α and $\tilde{\psi}_{\dot{\alpha}}$, where $\alpha = 1, 2$ is a $SU(2)_L$ fundamental index and $\dot{\alpha} = \dot{1}, \dot{2}$ is an $SU(2)_R$ index.

• Internal symmetries can be either global or local. I discussed this in 215a, and will review and extend the discussion here. Local symmetries are redundancies rather than symmetries, and lead to the basic forces: electromagnetism is associated with a $u(1)$ gauge symmetry; the weak interactions are associated with a $su(2)$ gauge symmetry; the strong interactions are associated with an $su(3)$ gauge symmetry; gravity is associated with general coordinate invariance gauge symmetry. I will discuss these things more in this class, but will omit gravity.

• As a reminder (from 215a) and illustration, consider $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$. There is a global $U(1)$ symmetry under $\psi \to e^{iq\alpha}\psi$. Here α is an arbitrary constant parameter, and q is a charge that I could set to 1 here, but kept it because sometimes we have multiple fields and their q's then could have to be adjusted based on the interactions. The Fermion ψ is a 4-component Dirac Fermion if $m \neq 0$. If $m = 0$, it could be one left-handed chiral Fermion, or a right-handed chiral Fermion, or it could be the original Dirac Fermion that consists of both. If it's both, then $U(1)$ becomes $U(1)_L \times U(1)_R$, since we can rotate the two chiral parts differently. If there are N Dirac Fermions of the same non-zero mass, then the global symmetry is $U(1)_V \times SU(N)$: the $\mathcal{L}$ is invariant under $\psi^i \to U^i_j \psi^j$ where $i = 1 \ldots N$ is a flavor index and U is a constant, unitary, $N \times N$ matrix. Recall $SU(N)$ is the symmetry group if we restrict to U with $\det U = 1$. If there are N massless Dirac Fermions, then the global symmetry is $U(1)_V \times U(1)_A \times SU(N)_L \times SU(N)_R$, where $U(1)_V$ is the part that acts on Dirac Fermions, and $U(1)_A$ acts with γ^5, e.g. the associated current is $\sim \bar{\psi}\gamma^5\gamma^\mu\psi$. Later on, we will discuss variants with gauge fields, and we will see there that the classical $U(1)_A$ global symmetry is violated by a quantum effect; this is called the ABJ anomaly.

• Briefly review group theory for continuous Lie groups G. We take G to be compact (e.g. $SU(2)$ rather than $SL(2)$ or $SU(1,1)$). Consider $g \in G = \exp(i\sum_{a=1}^{|G|}\phi_a T^a)$ and the group multiplication rule becomes commutation relations for the generators: $[T^a, T^b] = if^{abc}T^c$. For the case of $SU(2)$, $a = 1,2,3$ and $f^{abc} = \epsilon^{abc}$. We know this group well from the rotation group and angular momentum: $T^a = J^a/\hbar$. Symmetries show up in that our objects (states or operators) form *representations*. For $U(1)$, the representation is labeled by the charge q_e. For $SU(2)$, it is labelled by the analog of the spin j in the rotation group: $j \in \frac{1}{2}\mathbf{Z}$ labels a $(2j+1)$ dimensional representation, where $T^{a=3}$ is diagonal and runs from $\frac{j}{2}, \frac{j}{2}-1, \ldots -\frac{j}{2}$. The fundamental, **2** dimensional representation of $SU(2)$ has $T^a = \frac{1}{2}\sigma^a$, with σ^a the Pauli matrices. For $SU(N)$, the fundamental representation is that g are unitary $N\times N$ matrices with $\det g = 1$, and thus T^a are Hermitian traceless $N\times N$ matrices, and thus $|SU(N)| = N^2-1$. Every group also has the adjoint representation, which is $|G|$ dimensional, with $(T^a)_{bc} \to -if^{abc}$, i.e. $T^a|T^b\rangle = -if^{abc}|T^c\rangle = |[T^b, T^a]\rangle$; it follows from the Jacobi identity for triple commutators that this satisfies $[T^a, T^b] = if^{abc}T^c$. For $SU(2)$, the adjoint is $j = 1$, so e.g. $T^3 = \mathrm{diag}(1, 0, -1)$ in the $m = 1, 0, -1$ basis.

For any G, the generators in representation r have $\mathrm{Tr}(T_r^a, T_r^b) = T_2(r)\delta^{ab}$, where $T_2(r)$ is called the quadratic index of the representation. Also $\sum_{a=1}^{G} T_r^a T_r^a = C_2(r)\mathbf{1}_{|r|\times|r|}$, where $C_2(r)$ is the quadratic Casimir of the representation. Comparing gives $C_2(r) = |G|T_2(r)/|r|$. It is common to normalize the T^a as in $SU(2)$, so $T_2(fund) = \frac{1}{2}$. It is also a common notation to instead normalize the T^a such that $T_2(fund) = 1$, so if you need to track down factors of 2 in some reference, double check their T^a normalization conventions.

• The conserved charges in a theory with symmetry group G, either global or local, are in the adjoint representation of G. For $U(1)$, the adjoint is charge neutral; this is the statement that the charge Q operator measures charge, but does not carry its own charge since $[Q, Q] = 0$. The charges for general G, in a local theory, are $Q^a = \int d^3x J^{a,0}$ where a is the adjoint index and 0 is a Lorentz index, where $\partial_\mu J^{a,\mu} = 0$.

Global currents $J^{a,\mu}$ can be coupled to background gauge fields $A_\mu^a(x)$, which are also in the adjoint of G, via $\mathcal{L} \supset -\mathrm{Tr}A_\mu J^\mu + O(A^2)$. The backgrounds are an example of the sources that we introduce for every operator, to be able to compute correlation functions by taking functional derivatives of the effective action. For local, gauge symmetries the

$A^a_\mu(x)$ are dynamical (vs background) fields that we quantize, via canonical quantization or by functionally integrating over them in the path integral. For global symmetries, on the other hand, they are optional sources that we can imagine controlling with a some dial on a machine, like subjecting a system in the lab to an electric or magnetic field that the experimenter can control. We will see that non-zero $A^a_\mu(x)$ backgrounds require covariant derivatives $\partial_\mu \to D_\mu$ involving the gauge field $A^a_\mu(x)$. This is similar to the ∇_μ covariant derivatives of GR, but with the connection $\Gamma^\mu_{\rho\sigma}$ replaced with $-iA^aT^a$. In particular, current conservation equation becomes $D^\mu J^\mu$ where J^μ is in the adjoint and there are some commutator terms $\sim f^a_{bc}$ hidden in the D^μ – details will be given soon– and associated with that there are some higher order in $A^a(x)$ terms needed in the action. Non-abelian theories are non-linear.

We can likewise consider modifying the spacetime metric to some background metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. We know from GR that the metric is really dynamical, and presumably should be quantized. For low energies and small spacetime curvatures we can treat the metric as approximately fixed, rather than dynamically fluctuating, and usually we take $g_{\mu\nu} \to \eta_{\mu\nu}$. But we could also imagine somehow changing the metric to $g_{\mu\nu}$, which acts as a source for the energy momentum tensor $T^{\mu\nu}$: $\frac{\delta}{\sqrt{|g|}\delta g_{\mu\nu}} \to \frac{1}{2}T^{\mu\nu}$, with $\nabla_\mu T^{\mu\nu} = 0$.

• We now consider local, gauge symmetry, e.g. $\psi^i \to U^i_j(x)\psi^j$. The A^a_μ gauge fields are then required, dynamical fields, which we need to quantize, rather than optional background sources. For example, the strong nuclear force is associated with a dynamical, $su(3)$ gauge theory, and there are 8 associated gluon force carriers $A^{a,\mu}$, $a = 1 \dots |su(3)|$ with $|su(3)| = 8$. They are similar to the photon – massless, with two polarizations – in the perturbative analysis that is appropriate for UV processes. But unlike the photon the gluons have non-linear self-interactions, which leads to confinement in the IR. There is an unclaimed $\$10^6$ Clay prize for satisfactorily showing how this happens (there are many known ways to see it that are insufficient to claim the prize, including doing the functional integral by making spacetime a lattice and having a computer do the sum).

• Recall gauge theory for the case of QED. We take the Dirac Fermion ψ above and modify its $U(1)$ global symmetry to make it a local $u(1)$ "symmetry": $\psi \to e^{-iq\alpha(x)}\psi$. The derivatives have to be replaced with covariant derivatives, such that $D_\mu\psi \to e^{-iq\alpha(x)}D_\mu\psi$, and then $D_\mu\psi = (\partial_\mu + iqA_\mu)\psi$ and $A_\mu \to A_\mu + \partial_\mu\alpha$. No physics can depend on the choice of α so this is not just a symmetry, but really a redundancy of our description; it looks almost like a figment of our imagination, but it is needed for locality and has physical manifestations. Anyway,

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi, \qquad D_\mu \psi = \partial_\mu + iqA_\mu.$$

Sign check: $\mathcal{L} \supset -A_\mu J^\mu$ in our mostly minus convention, and here $J^\mu = q\bar{\psi}\gamma^\mu \psi$. Gauge configurations differing by a gauge transformations are equivalent, and in the functional integral we only integrate over physically distinct, inequivalent configurations (integrating over the gauge orbit gives an infinite factor):

$$Z[sources] = \int [d\psi][dA^\mu / gauge] e^{iS/\hbar}.$$

• There is another Lorentz and gauge invariant term that we can add to the QED Lagrangian density, $\sim \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \sim \vec{E} \cdot \vec{B}$. Note that this term violates P (since $\vec{E}$ is a vector and $\vec{B}$ is an axial pseudovector) and T (since $\vec{E}$ is even and $\vec{B}$ is odd). Also note that, unlike $F_{\mu\nu} F^{\mu\nu}$, we do not need to use the metric to contract the indices of $\sim \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$; indeed, this term is a topological quantity (it doesn't care if the metric of spacetime is flat, that of a black hole, FRW, etc).