215c, 4/29/20 Lecture outline.

⋆ **Week 5 reading: Tong chapter 2.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Last time, started to discuss the gauge field propagator. As in QED, we can see it from the path integral perspective by considering the $A^a_\mu A^b_\nu$ terms in $\mathcal{L}$ and the Gaussian integral leads to the inverse of the differential operator that acts on them. As in QED, gauge invariance means that the differential operator has zero modes from pure gauge configurations, so need care in inverting it. A procedure is to fix a gauge and then one can check that the results in the end are gauge invariant.

Naively doing the same procedure in non-Abelian gauge theories does not work. This is related to the fact that the gauge fields are charged, in the adjoint representation, and the currents are covariantly conserved. The upshot is that we have to add some new fictitious fields, called ghosts, to subtract off the unphysical polarizations in loops and make things work. Feynman and Bryce DeWitt (independently) first noticed this in the context of writing down a quantum theory for gravitons in the early '60s. They realized that Yang-Mills theory was a good toy model, exhibiting similar non-linearities as gravity, and they initially just worked out what the ghosts had to be in order to get sensible, gauge invariant physics at the end. This was later systematized by Fadeev and Popov.

Picture gauge field space, and the physically equivalent gauge orbits. The path integral should integrate over physically distinct configurations, i.e. it should only count one gauge field in each orbit. The gauge fixing condition should fix to a slice in this space that intersects each orbit once. If we instead integrate over the gauge equivalent orbits, we get an extra factor of the volume of the gauge group and then we can divide by that overall multiplicative factor (it's just overall multiplicative by gauge invariance). Let's first consider the case of QED, where the gauge orbit is $A_\mu \to A_\mu + \partial_\mu \omega$ for any ω. We can gauge fix by imposing some $F(A) = 0$ which slices through each gauge orbit once, e.g. taking $F(A) = \partial_\mu A^\mu$. Alternatively, we can impose $F(A) = f(x)$ and then introduce $\int [df] G[f]$ into the path integral, which is just a normalization factor. We like to take $G[f] = \exp(-\frac{i}{2\xi}\int f^2)$ and then, when we gauge fix to $f = \partial_\mu A^\mu$, we get the photon propagator $i(-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2})/(p^2 + i\epsilon)$ and the results are independent of ξ at the end of the day ($\xi = 1$ is Feynman gauge and $\xi = 0$ is Landau gauge. The non-gauge fixed theory is like $\xi \to \infty$, which looks singular.)

We will repeat the analog of this for non-Abelian gauge theory, but the argument was a little sloppy - in a way that does not matter for abelian theories but introduces ghosts for non-Abelian theories. The sloppy point was, when we impose $F(A) = f$, we should think about this as introducing a $\delta[F(A) - f]$ and we need the functional analog of $\delta(f(x)) = \frac{1}{|f'(x)|}\delta(x - x_0)$. We should insert $1 = \int d\omega^a(x)\delta(F(A^\omega) - f)\det(\frac{\delta F(A^\omega)}{\delta\omega})$ where $U(x) = \exp(iT^a\omega^a(x)) \in G$ are the gauge transformation group elements, and $d\omega^a(x)$ is some group invariant integration measure in the Lie algebra (called the Haar measure). Group invariant means that we can do a change of variables $U(x) \to gU(x)$ where g is any group element and we get the same thing (e.g. if we're integrating over the Euler angles we can shift the origin). We then get

$$\int d\omega^a \int [dA_\mu^a] \int df \delta[F(A) - f] \det(\frac{\delta F(A^\omega)}{\delta\omega}) e^{iS + \frac{i}{2\xi}\int f^2}$$

The delta function was introduced for getting the gauge slice from $[dA]$ but we can instead use it to do $\int[df]$. The$\int d\omega^a$ factors out as an overall normalization constant. Introducing sources for A_μ^a and doing that path integral gives the propagator $i\delta^{ab}(-g^{\mu\nu} + (1 - \xi)\frac{p^\mu p^\nu}{p^2})/(p^2 + i\epsilon)$

The determinant is the Fadeev Popov determinant, and it will be obtained via a gaussian integral of fictitious scalar fields. Because the determinant does not have a square-root, these are scalar fields, and they are in the adjoint of the group. Because the determinant is in the numerator, the scalar fields are grassmann, anticommuting – so they contribute with a minus sign in loops. That minus sign, and the fact that it is two real fields in the adjoint, are related to the fact that they're subtracting off two unphysical polarizations of the gauge field.

- The cubic and quartic A_μ terms in $\mathcal{L}_{YM}$ lead to cubic and quartic Feynman diagrams. The diagram with gauge fields with labels (a, μ, k), (b, ν, p), (c, ρ, q) contributes $gf^{abc}[g^{\mu\nu}(k - p)^\rho + g^{\nu\rho}(p - q)^\mu + g^{\rho\mu}(q - k)^\nu]$. The quartic vertex contributes $-ig^2[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + 2perms]$. Let's illustrate by starting to compute the one-loop correction to the gluon propagator from the diagram using two cubic vertices. Let the external gluons have labels (a, μ, q) and $(b, \nu, -q)$ and the internal gluons have (c, ρ, p) and $(d, \sigma, q + p)$. Get (the $\frac{1}{2}$ is a symmetry factor)

$$\tfrac{1}{2}g^2 f^{acd} f^{bcd} \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2} \frac{-i}{(p + q)^2}[g^{\mu\rho}(q - p)^\sigma + 2terms][\delta_\rho^\nu(p - q)_\sigma + 2terms].$$

Recall that $\sum_{a=1}^{|G|} T^a(r)T^a(r) = C(r)\mathbf{1}_{|r|\times|r|}$, with $C(r)$ the quadratic Casimir of the representation, and for the adjoint representation $T^a \to -if^{abc}$ and this gives $\sum_{ab} f^{abc}f^{abd} = C(adj)\delta^{cd}$. For $SU(2)$ using $f^{abc} = \epsilon^{abc}$ gives $C(adj) = 2$ ($= j(j+1)$ with $j = 1$). More generally, $C(adj, SU(N)) = N$.

We can combine the denominators using the Feynman trick, giving $\int_0^1 dx \frac{1}{(x(p+q)^2+(1-x)p^2)^2}$, and we go to Euclidean d dimensions and do the $\int d^dp$ integrals using the standard integral formulae e.g. $\int \frac{d^dp_E}{(2\pi)^d}\frac{1}{(p_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac{1}{2}d)}{\Gamma(n)}\Delta^{\frac{1}{2}d-n}$.. The result is a mess and does not have the correct Lorentz structure. Adding the loop with the quartic vertex does not help. This problem is solved by adding the ghosts. Adding the ghost loop gives a reasonable transverse result. It turns out to be that the three diagrams sum to $i(q^2g^{\mu\nu} - q^\mu q^\nu)\delta^{ab}C_2(adj)\frac{g^2}{(4\pi)^2}[(\frac{13}{6} - \frac{\xi}{2})\Gamma(2 - \frac{1}{2}d) + \ldots]$. There are also tadpole type diagrams, with a gauge field tail connecting to a blob head – such diagrams must give zero because they can be interpreted as a one-point function correlator of the gauge field current, which must vanish both by Lorentz invariance (as in QED), and also gauge invariance. The ξ will cancel in the end for physical processes.

• Let's also consider the leading, tree-level contribution with external sources in representations r of G. For the case of QED, we replace r with the charge q and the interaction vertex the coupling of $A_\mu j^\mu$ gives a factor of ieq (recall the A_μ sign change). We saw in the Fall that the t-channel photon exchange Feynman diagram gives the Coulomb potential, here weighted by e^2q^2. The only difference in the non-Abelian case is we replace $ieq \to igT^a(r)$. For the case of the adjoint representation, recall that $T^a \to -if^{abc}$ and this is consistent with how the gluon couples in the 3-gluon vertex. We recover the same Coulomb potential from the leading-order, tree-level diagram, with $e^2q^2 \to g^2\sum_a(T^a(r)T^a(r))$, i.e. the Coulomb potential is weighted by $g^2C(r)$. Aside: $\mathrm{Tr}_r(T^aT^b) = T_2(r)\delta^{ab}$ and comparing gives $C(r) = |G|T_2(r)/|r|$. The Casimir of the fundamental of $SU(N_c)$ is $C(fund) = (N_c^2-1)/2N_c$ (e.g. for $N_c = 2$ this is $j(j+1)$ with $j = \frac{1}{2}$) while that of the adjoint is $C(adj) = T_2(adj) = N_c$.

Another way to understand the need for ghosts in non-Abelian gauge theories (see e.g. Peskin and Schroeder section 16.1) is to consider the one-loop correction to this tree-level process. In QED, the intermediate photon propagator is replaced with its one-loop correction, with a fermion loop. The optical theorem relates the imaginary part of this loop diagram to the square of the tree-level diagram where we cut across the loop. In Yang-Mills, we have similar diagrams where we replace the fermion loop with a gluon loop, and we can think of the tree-level process as $r \times \bar{r} \to \gamma \to \gamma\gamma$ where γ now refers to the gauge field (aka gluon). The puzzle is that all four gluon polarizations run in the virtual gluon loop, whereas the external gluons should be on-shell and thus only have two polarizations. The fix will be to add ghosts that also run in the loop, and properly subtract off the unphysical polarization contributions.

- We now put in the FP determinant via S_{ghost} term for adjoint valued, anti-commuting scalar ghost fields c^a. (There is a remaining symmetry, found by BRST (Becchi, Rouet, Stora, and independently Tyutin), such that physical states = the co-homology of an operator Q_{BRST}, which can be used to prove that the physical states are unitarily ghost-free.) The ξ gauge fixing is associated with $F(A) = (\partial_\mu A^\mu)^a$ (yes, they're ordinary rather than covariant derivatives). An infinitesimal gauge transformation gives $\delta F^a = \partial^\mu(f^{abc}\delta\omega^b A^c_\mu(x) - \partial_\mu \delta\omega^a(x))$ so $\frac{\delta F^a(x)}{\delta\omega^b(y)} = \partial^\mu([\delta^{ab}\partial_\mu - f^{abc}A^c_\mu]\delta(x-y))$ and

$$\Delta_{FP} = (const)\det(\partial^\mu D_\mu) = (const)\int dc^a d\bar{c}^a e^{iS_{ghost}}, \qquad S_{ghost} = \int d^4x \mathrm{Tr}(\partial_\mu \bar{c} D^\mu c).$$

The ghost propagator is $\frac{i\delta^{ab}}{p^2+i\epsilon}$ (and loops have a minus sign since it's anticommuting), and there is a ghost vertex with the gauge field, weighted by $gf^{abc}p_\mu$ (where the incoming ghost has index a, the gauge field has index c and μ, and the outgoing ghost has index b and momentum p_μ). So the ghost loop contribution to the gluon propagator is

$$(-1)g^2 f^{acd} f^{bdc} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2} \frac{i}{(p+q)^2} (p+q)^\mu p^\nu$$

$$\to ig^2(4\pi)^{d/2} C_2(adj)\delta^{ab} \int_0^1 dx \Delta^{d/2-2}(-\tfrac{1}{2}\Gamma(1-d/2)g^{\mu\nu}q^2 + \Gamma(2-d/2)q^\mu q^\nu)x(1-x),$$

which adds to the other diagrams to cancel an unwanted pole at $d=2$ and give the correct transverse Lorentz structure.