215c, 5/4/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 5 reading: Tong chapter 2.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Last time: we need to gauge fix to write down a sensible gauge field propagator. Get $\delta[F(A)-f]\det\frac{\delta F(A^\omega)}{\delta\omega}e^{-\frac{i}{2\xi}\int d^4xf^2}$ which leads to $\mathcal{L}_{g.f.}=-\frac{1}{2\xi}F^aF^a$ and $\mathcal{L}_{ghost}=-\bar{c}^a\frac{\partial F^a}{\partial A^b_\mu}(D_\mu c)^b$. For $F=\partial_\mu A^\mu$ gives $\mathcal{L}_{g.f.}=-\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2$ and $\mathcal{L}_{ghosts}=\partial^\mu\bar{c}^a(D_\mu c)^a$ with $(D_\mu c)^a=(\delta^{ab}\partial_\mu-gf^{abc}A^c_\mu)c^b$. The gauge field propagator is then $i\delta^{ab}(-g^{\mu\nu}+(1-\xi)\frac{p^\mu p^\nu}{p^2})/(p^2+i\epsilon)$, the ghost propagator is $\frac{i\delta^{ab}}{p^2+i\epsilon}$, and there is a ghost vertex with the gauge field, weighted by $gf^{abc}p_\mu$ (where the incoming ghost has index a, the gauge field has index c and μ, and the outgoing ghost has index b and momentum p_μ). As mentioned last time, the gauge field propagator gets corrected by two diagrams with internal gauge fields, and each diagram, and the sum, does not have the required behavior: it should be transverse, i.e. $\sim(q^2g_{\mu\nu}-q_\mu q_\nu)\Pi(q^2)$, and we expect from experience that $\Pi(q^2)$ will have to be regulated and that in dim-reg will give a $\Gamma(2-d/2)$. Instead the two diagrams are non-transverse, and have additional $\Gamma(1-d/2)$ terms. The ghost loop fixes these problems. The ghost loop contribution to the gluon propagator is

$$(-1)g^2f^{acd}f^{bdc}\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2}\frac{i}{(p+q)^2}(p+q)^\mu p^\nu$$

$$\to ig^2(4\pi)^{d/2}C_2(adj)\delta^{ab}\int_0^1 dx\Delta^{d/2-2}(-\tfrac{1}{2}\Gamma(1-d/2)g^{\mu\nu}q^2+\Gamma(2-d/2)q^\mu q^\nu)x(1-x),$$

with $\Delta=-x(1-x)q^2$. This adds to the other diagrams to cancel an unwanted pole at $d=2$ and give the correct transverse Lorentz structure.

• Unitarity, e.g. if there is an S-matrix then $S_{fi}=\delta_{fi}+i(2\pi)^4\delta(p_i-p_f)T_{fi}$ is unitary, and then $T_{fi}-T^*_{if}=i\sum_n T^*_{nf}T_{ni}(2\pi)^4\delta(p_i-p_n)$ where the sum is over physical states. So the issue, discussed last time, is that if we cut an internal gauge field propagator and replace it by its imaginary part, corresponding to the LHS, then we want to get only physical states – but we do not. For example, the imaginary part of the gauge field propagator in Feynman gauge ($\xi=1$) is $-2\pi\delta^{ab}g_{\mu\nu}\delta(k^2)\Theta(k^0)$, which looks good for external gauge fields except that δ^{ab} includes all polarizations. The way out in QED is that the photon couples to a conserved current, so if we replace $\epsilon^\mu\to k^\mu$ for an external photon we get zero anyway $k_\mu\mathcal{A}^{\mu\nu}=0$. This does not generalize to the non-Abelian case and instead we add the additional ghost-loop diagram to the gauge field propagator, which cancels off the longitudinal polarization contributions.

There is a nice way to prove that the ghosts do the right thing and that the resulting theory is unitary. It uses what is known as BRST quantization, (Becchi, Rouet, Stora, and independently Tyutin). It introduces a Fermionic BRST symmetry Q, which acts on the physical fields like a gauge transformation with gauge parameter $\omega^a \to c^a$, e.g. $i[Q, A_\mu^a] = (D_\mu c)^a$. It acts on the ghost c^a as $i\{Q, c^a\} = -gf^{abc}c^b c^c$, and $i\{Q, \bar{c}^a\} = b^a$ where b^a enters by replacing $S_{g.f}$ with $\mathcal{L}_b = b^a \partial_\mu A^{\mu a} + \frac{1}{2}\xi b^a b^a$ (this last step is a standard trick, sometimes called the Hubbard-Stratonovich transformation and sometimes called a Legendre transformation). Finally $Qb^a = 0$. It can be shown that $Q^2 = 0$. There is a similar $\bar{Q}$ symmetry that acts on the physical fields like a gauge transformation where the gauge parameter is $\bar{c}^a$ and a more complicated action on b^a and c^a, and which satisfies $\{Q, \bar{Q}\} = \bar{Q}^2 = 0$. The gauge fixing and ghost terms can be obtained by writing down the most general theory consistent with the Q and $\bar{Q}$ symmetry with ghost number zero. The FP procedure does not work properly when the gauge fixing condition $F^a = 0$ is not linear in the gauge field, but the BRST procedure always works.

The physical states are in the cohomology of Q and $\bar{Q}$. It's a long story, which could be a topic for the final presenation.

• We can also consider a Fermion in representation r_f, with $\mathcal{L} \supset \bar{\psi} i \not{D} \psi$ which leads to a cubic vertex with one gauge field and two Fermions, weighted by $ig\gamma^\mu T^a_{r_f}$. We could also consider a scalar in representation r_s, which yields a 3-point vertex with weight $igT^a(p^\mu + q^\mu)$ where p and q are the incoming momenta of the scalars. There is also the quartic seagull vertex $ig^2 g^{\mu\nu}\{T^a, T^b\}$.

• Let's write the Lagrangian as having terms

$$\mathcal{L} \supset -\frac{1}{4}Z_3[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\frac{Z_{1,YM}}{Z_3} f^{abc} A_\mu^b A_\nu^c]^2 + \tilde{Z}_2 \bar{c}^a[\delta^{ab}\partial^2 - \tilde{g}(\tilde{Z}_1/\tilde{Z}_2) f^{abc} A_\mu^c \partial^\mu] c^b - \frac{1}{2\xi}(\partial_\mu A^{\mu a})^2$$

where we will want our renormalization scheme such that $g = \tilde{g}$ and $gZ_{1,YM}/Z_3 = \tilde{g}\tilde{Z}_1/\tilde{Z}_2$, with analogous relations if we add matter fields, so that the gauge field couples universally with the same gauge coupling g. The bare quantities are $A^a_{B,\mu} = \sqrt{Z_3} A^a_\mu$, $c^a_B = \sqrt{\tilde{Z}_2} c^a$, $g_B = gZ_{1,YM}/Z_3^{3/2} = \tilde{g}\tilde{Z}_1/\tilde{Z}_2 Z_3^{1/2}$. We can re-write it in terms of the original Lagrangian for the physical fields, and counter-terms

$$\mathcal{L}_{c.t.} = -\frac{1}{4}\delta_3[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a]^2 - \frac{1}{2}\delta_{A^3} g(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) f^{abc} A^{b,\mu} A^{c,\nu} -$$

$$\frac{1}{4} g^2 \delta_{A^4} f^{abc} f^{aef} A_\mu^b A_\nu^c A^{e,\mu} A^{f,\nu} + \tilde{\delta}_2 \bar{c}^a \partial^2 c_a - \tilde{\delta}_1 f^{abc} \bar{c}^a A_\mu^b \partial_\mu c^c \ldots$$

where $\delta \equiv Z-1$, $\delta_{A^3}=Z_{1,YM}-1$, and $\delta_{A^4}=(Z_{1,YM}^2 Z_3^{-1}-1)$ etc . Z_3 gets contributions from loop (gauge field + ghosts+matter) corrections to the 2-gauge field propagator, $Z_{1,YM}$ gets contributions from the loop corrections to the 3-gauge field vertex, $Z_{1,YM}Z_3^{-1}$ gets contributions from the loop corrections to the four-gauge field vertex. For example, in $d=4-\epsilon$ dimensions, the three one-loop contributions to the gauge field propagator (in pure Yang-Mills – there is an additional contribution from matter fields running in the loop) leads to $\delta_3=\frac{1}{\epsilon}\frac{g^2}{16\pi^2}(\frac{10}{3}+1-\xi)C(adj)+\ldots$, and $\delta_{A^3}=\frac{1}{\epsilon}\frac{g^2}{16\pi^2}(\frac{4}{3}+\frac{3}{2}(1-\xi))C(adj)+\ldots$ and $\delta_{A^4}=\frac{1}{\epsilon}\frac{g^2}{16\pi^2}(-\frac{2}{3}+2(1-\xi))C(adj)+\ldots$ Of course, the gauge parameter ξ must drop out at the end for physical quantities.

We now write $g_B=g\mu^{\epsilon/2}Z_{1,YM}/Z_3^{3/2}=g\mu^{\epsilon/2}(1+\delta_{A^3}-\frac{3}{3}\delta_3)+\mathcal{O}(g^4)$ (or any of the other relations, expressing universality of the gauge coupling), and require that the bare quantities are μ independent. This gives $\beta(g)=\frac{dg}{d\ln\mu}=-\frac{1}{2}\epsilon g-\frac{d}{d\ln\mu}(\delta_{A^3}-\frac{3}{2}\delta_3)+\ldots$ with $\delta_{A^3}-\frac{3}{2}\delta_3=\frac{1}{\epsilon}\frac{g^2}{16\pi^2}C(adj)(-\frac{11}{3})+\ldots$. It's a cross check that ξ indeed dropped out. We use the fact that each δ only depends on μ via the renormalized g to replace $-\frac{d}{d\ln\mu}\to-\frac{d\ln g}{d\ln\mu}\frac{d}{d\ln g}\to\frac{\epsilon}{2}\frac{d}{d\ln g}$. The upshot for $\epsilon\to 0$ is $\beta(g)=-\frac{g^3}{48\pi^2}(11C(adj)-4T(r_{DF})-T(r_{CS}))+\mathcal{O}(g^5)$ where the last expression includes the effect of Fermions in representation r_{DF} and scalars in representation r_{CS}. Here DF stands for Dirac Fermion and we should divide the contribution to the beta function in half if the Fermions are chiral; likewise CS stands for complex scalars and we should divide the contribution in half if the scalars are real.

• There is an alternative method to the direct assault on the gauge field propagator and other contributions, which is cleaner, requires fewer diagrams, and is conceptually interesting. It is called background field gauge. It is like having your cake and eating it too. On the one hand, we need to break gauge invariance, fixing a gauge, to write the gauge field propagator. Then we have to wait until the dust settles in the calculation to see that the final result is gauge invariant, e.g. the cancellation of the ξ parameter mentioned above. Background field gauge is a way to break gauge invariance, but still have manifest background gauge invariance in the calculation. Global currents can be coupled to background gauge fields sources. For a gauge symmetry, there are instead dynamical gauge fields that we integrate over in the path integral. But we can still couple a global component of the gauge symmetry to a background source. There are then two types of gauge transformation: that of the dynamical gauge field, and that of the background source. We can choose gauge fixing such that they violate gauge invariance but preserve the sum of gauge plus background gauge invariance.