215c, 5/11/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 7 reading: Tong chapter 2.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• The fine-structure constant in QED is usually defined as $\alpha = e^2/4\pi\hbar c$. Each additional loop comes with a factor of $e^2 \int \frac{d^4p}{(2\pi)^4}$ and $\int d^Dp$ gives a solid angle factor of $\Omega_P = \frac{2\pi^{\frac{D}{2}}}{\Gamma(D/2)}$ which in $D=4$ is the unit S^3 area $2\pi^2$, so loops are weighted by $e^2/8\pi^2 = \alpha/2\pi \equiv \hat{\alpha}$. The $1/2\pi$ factors help with perturbation theory. Let's likewise define $\hat{\alpha} = g^2/8\pi^2$ for non-Abelian gauge theories.

• Write the one-loop beta function as $\frac{d}{d\ln\mu}\hat{\alpha}^{-1} = -16\pi^2 g^{-3}\beta(g) \equiv b_1$, which integrates to $e^{-8\pi^2/g^2(\mu)} = (\frac{\Lambda}{\mu})^{b_1}$, with $b_1 = \frac{1}{3}(11C(adj) - 4T(r_{DF}) - T(r_{CS}))$. For example, for $SU(N_c)$ gauge field with N_f Dirac flavors in the fundamental this gives $b_1 = \frac{1}{3}(11N_c - 2N_f)$. The theory is asymptotically free if $N_f < \frac{11}{2}N_c$. Aside: with supersymmetry we add new fields: gauginos which are adjoint chiral Fermions (2-component, to match the two polarizations of the gauge fields) and complex scalars in the $N_c + \bar{N}_c$ to match the d.o.f. of the Fermion flavors, so $b_1 = \frac{1}{3}(11N_c - 2N_f - 2N_c - N_f) = 3N_c - N_f$, so the theory is asymptotically free if $N_f < 3N_c$.

• If we go to two loops, $\frac{d}{d\ln\mu}\hat{\alpha}^{-1} = -16\pi^2 g^{-3}\beta(g) \equiv b_1 + b_2\hat{\alpha}$, and it turns out that the 2-loop beta function coefficient b_2 is (e.g. for a theory without charged scalars) $b_2 = \frac{34}{3}C_A^2 - 4N_f C_{DF}T_{DF} - \frac{20}{3}N_f C_A T_{DF}$. Recall that $\sum^a T_r^a T_r^a = C(r)\mathbf{1}_{|r|}$ and $\mathrm{Tr}_r T_r^a T_r^b = \delta^{ab}T(r)$ and thus $C(r)|r| = |G|T(r)$. E.g. for $SU(N_c)$ with N_f fundamental flavors we set $C_A = N_c$, $T_fund = \frac{1}{2}$, so $C_{fund} = (N_c^2-1)/2N_c$ (as a check, for $N_c = 2$ this gives $3/4 = j(j+1)$ for $j = \frac{1}{2}$). To simplify the expressions, let's consider the case of large N_c (I will discuss other aspects of the large N limit later, time permitting), so $C_{fund} \approx \frac{1}{2}N_c$ and then $SU(N_c)$ with N_f Dirac flavors in the fundamental has $b_2 = \frac{34}{3}N_c^2 - \frac{13}{3}N_c N_f$. Note that, for N_f just below $\frac{11}{2}N_c$ the one-loop beta function is negative ($b_1 > 0$), and the two-loop beta function is positive ($b_2 < 0$), so there is a possibility of having $\beta = 0$, approximately where $b_1 + b_2\alpha = 0$ so $\hat{\alpha}_* \approx |b_1/b_2|$. In the large N_c and N_f limit, with the ratio fixed, we can tune b_1 to be $\mathcal{O}(N_c^0)$ and b_2 is $\mathcal{O}(N_c^2)$ so $\hat{\alpha}_* = \mathcal{O}(N^{-2})$. Note that in the large N_c and N_f limit, generically $b_n \sim N^n$, which exhibits a general fact about large N: the loop counting parameter is $\hat{\alpha}N$, so we need $\alpha \sim 1/N$ to hope for a good perturbative expansion. The fact that we can make $\hat{\alpha}_* N = \mathcal{O}(N^{-1})$ means that its easily perturbative in the large N limit.

Lattice gauge theory gives a way to determine the conformal window outside of perturbation theory, and this has been part of Prof. Julius Kuti's research program.

• If we plot the one-loop $\hat{\alpha}^{-1}$ as a function of $\ln \mu$, the one-loop running gives a straight line, with slope b_1; asymptotically free couplings have $b_1 > 0$ and non-asymptotically free ones have $b_1 < 0$. Let's consider the effect of a field of mass m. For energies $\mu > m$, the massive field contributes in the loop and there is some beta function $\beta_H(g)$. For energies $\mu < m$, the massive field decouples (this is not obvious in mass independent renormalization schemes, e.g. $\overline{MS}$), and the low-energy theory has beta function $\beta_L(g)$. The coupling g is of course continuous across the scale m, and this gives a threshold matching relation for the dynamical scale. To one-loop, this gives $(\frac{\Lambda_H}{m})^{b_{1,H}} = (\frac{\Lambda_L}{m})^{b_{1,L}}$, i.e. $\Lambda_L^{b_{1,L}} = m^{b_{1,L}-b_{1,H}}\Lambda_H^{b_{1,H}}$, e.g. for $SU(N_c)$ with N_f matter fields in the fundamental, if we give a mass to one flavor, then the slope changes from $b_{1,H} = \frac{1}{3}(11N_c - 2N_f)$ to $b_{1,L} = \frac{1}{3}(11N_c - 2(N_f - 1))$ and $\Lambda_L^{b_{1,L}} = m^{2/3}\Lambda_H^{b_{1,H}}$. Note that we need $\mu, m > \Lambda$ and thus $\Lambda_L > \Lambda_H$; this reflects the fact that, by decoupling a matter field, the low-energy theory is more asymptotically free, and thus more strongly coupled in the IR.

• Let's consider the SM: the gauge group is $SU(3)_C \times SU(2)_W \times U(1)_Y$ with three generations of chiral fermions in the $(3,2)_{1/3} + (\bar{3},1)_{-4/3} + (\bar{3},1)_{2/3} + (1,2)_{-1} + (1,1)_2$. The Higgs field is a complex scalar in the $(1,2)_1$. For $U(1)_Y$ the slope is $b_1 = -\frac{4}{3}\text{Tr}_{DF}Q^2 - \frac{1}{3}\text{Tr}_{CS}Q^2$. The slopes for $SU(3) \times SU(2) \times U(1)_Y$ are then found to be $(7, \frac{19}{6}, -\frac{41}{3})$. The three couplings approximately, but do not quite meet at $\mu \sim 10^{15}GeV$. This is some approximate evidence for grand unification, and the fact that they miss can be modified by adding additional matter fields at higher energy. E.g. in the MSSM get slopes $(3, -1, -33/5)$. and better convergence of the couplings at $M_{GUT?} \sim 10^{15}GeV$.

• QCD is asymptotically free in the UV, and strongly coupled in the IR. It does not RG flow in the IR to a CFT. Instead, a Fermion bilinear operator gets a vacuum expectation value, which spontaneously breaks (approximate) global symmetry, leading to the observed light pions. We will sketch the (sketchy) details soon.

• Let's first consider spontaneous symmetry breaking, aka (Coleman): secret symmetry. The "spontaneous" and "breaking" are potentially confusing, e.g. the symmetry is not broken at all, but only obscured by the groundstate.