215c, 5/18/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 8 reading: Tong chapter 5 up to WZW term.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

Also Coleman's Aspect of Symmetry book, chapter on secret symmetry.

• Consider $su(N_c)$ gauge theory with N_f Dirac fermions in the fundamental. Take the fermions to be massless – we will discuss adding mass terms soon. As discussed last time, we decompose the Dirac Fermions into left and right-handed pieces via $P_{L,R} = \frac{1}{2}(1 \pm \gamma^5)$; in the Dirac basis where $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then $\psi^{f,c} \to \begin{pmatrix} \psi^{f,c}_\alpha \\ \tilde\psi^{\dagger f,c}_{\dot\alpha} \end{pmatrix}$ where we choose to write the Fermions as both left-handed Weyl, $\psi^{f,c}_\alpha$ and $\tilde\psi_{\tilde f,c,\alpha}$ in the N_c and $\bar N_c$. Write the symmetry table, including the $\delta\mathcal{L} = m_{i\tilde j}\psi^i_\alpha \tilde\psi^{\tilde j}_\beta \epsilon^{\alpha\beta} + h.c.$ (note that the masses can be complex).

For $N_f \geq \frac{11}{2}N_c$, the theory is IR free and needs a UV cutoff or completion. For $N_f \leq N_c$, the theory is asymptotically free in the UV, and the gauge coupling g is relevant in the IR, flowing to stronger coupling. It can either flow to a zero of $\beta(g)$, which is believed to happen for N_f in the conformal window $N_{f,*} < N_f < \frac{11}{2}N_c$, or it flows to stronger coupling and something else happens. A possibility, which happens in real-world QCD, is spontaneous symmetry breaking: $\langle \psi^{f,c}_\alpha \tilde\psi^\beta_{\tilde f,d}\rangle = -\sigma\delta^c_d\delta^\alpha_\beta\delta^f_{\tilde g}$ where the usually-suppressed indices are written out to emphasize that this vev does not break $su(N_c)$ gauge or $SU(2)_L$ Lorentz symmetry, but it does break $SU(N_f) \times SU(N_f) \times U(1)_A \to SU(N_f)_D$. The $\sigma \sim \Lambda^3$. In perturbation theory, gauge field exchange between the Fermions leads to an attractive force in the channel where $N_c \times \bar N_c \to 1$ and a repulsive force in the channel where $N_c \times \bar N_c \to (N_c^2 - 1)$. There is also a perturbative interaction, already at tree-level and $\mathcal{O} * g^2$), where $\psi \leftrightarrow \psi + (\psi\tilde\psi)$ and this gives some picture that the vacuum can develop a $\psi\tilde\psi$ condensate. We can give a Landau-Ginsberg picture where, for small g, the effective potential for the gauge and Lorentz invariant order parameter $\mathcal{O}^i_{\tilde j} = \psi^i\tilde\psi_{\tilde j}$ has its minimum at the origin, but for large enough g there is a phase transition (the order of the transition is an interesting question, that has been studied on the lattice) to a Mexican hat potential that leads to $\langle\mathcal{O}\rangle \neq 0$, spontaneously breaking $G \to H$. As a result, the low-energy theory contains NGBs in $G/H \cong SU(N_f)_{L-R}$; these are the pions, and the usual notation is to write $U(x) \in SU(N_f) = \exp(\frac{2i}{f_\pi}\pi(x))$ with $\pi(x) \equiv \pi^a(x)T^a$. E.g. for $SU(2)$ the π^a are the Euler angles, which live in a compact space and parameterize a $S^3 \cong SU(2)$.

The broken $SU(N_f)_L \times SU(N_f)_R$ act on U as $U \to L^\dagger U_R$ and using the unbroken $SU(N_f)_D$ we can locally rotate $U \to 1$ and see that is preserved if $L = R$. Note that

$\mathrm{Tr}(U^\dagger\partial_\mu U)=0$ because the generators are traceless, and $U^\dagger\partial_\mu U=-\partial U^\dagger U$, so there is a unique kinetic term that respects the symmetry (called the chiral Lagrangian):

$$\mathcal{L}_2=\frac{f_\pi^2}{4}\mathrm{Tr}(\partial^\mu U^\dagger\partial_\mu U)=\tfrac{1}{2}\partial^\mu\pi^a\partial_\mu\pi^a-\frac{2}{3f_\pi^2}\mathrm{Tr}(\pi^2(\partial\pi)^2-(\pi\partial\pi)^2)+\ldots.$$

This describes the massless pions, and their derivative interactions. This is the low-energy effective field theory for the spontaneously broken phase, and the theory can be treated as a LEEFT with a cutoff at $\sim f_\pi\sim v\sim\Lambda$. We can see how aspects of the original, microscopic theory show up in the LEEFT dual.

The $SU(N_f)_D$ global symmetry is manifest and unbroken in the LEEFT. The separate $SU(N_f)_L$ and $SU(N_f)_R$ secret symmetries act as $U\to L^\dagger UR$ and are thus realized as shifts of the NGBs. The $SU(N_f)_L$ current acts as $\delta_L U\approx -i\alpha^aT^aU\approx-\frac{1}{2}f_\pi\partial_\mu\pi^a$, which fits with the fact that these currents do not annihilate the vacuum but instead act on it to create the NBGs, $\langle\pi^b(p)|J^a_{L,\mu}(x)|0\rangle=i\frac{f_\pi}{2}\delta^{ab}p_\mu e^{ipx}$. Likewise $J^a_{R,\mu}\approx+\frac{1}{2}f_\pi\partial_\mu\pi^a$ so the diagonal sum is unbroken.

Parity P takes $\vec{x}\to-\vec{x}$ and thus $\gamma_5\to-\gamma_5$ and thus exchanges $L\leftrightarrow R$. It thus takes $U\to U^*$ so $\pi^a\to-\pi^a$. This fits with $\partial_\mu\pi^a\sim J^a_{L-R,\mu}$. This shows that pions transform as parity odd pseudoscalars, which fits with observation (based on which decays are allowed vs suppressed).

• If we add mass terms $\delta_{\mathcal{L}}=m_{i\tilde{j}}\psi^i\tilde{\psi}^{\tilde{j}}+h.c.$, we explicitly break $SU(N_f)_L$ and $SU(N_f)_R$ because the masses are in the $(\bar{N}_f,N_f)$. This shows up in the LEEFT as mass terms for the NGBs: $\delta_{\mathcal{L}}=\frac{1}{2}\sigma\mathrm{Tr}mU+h.c.\to-\frac{\sigma}{f_\pi^2}\mathrm{Tr}(m+m^\dagger)\pi^2+\ldots$. Note that the mass-squared of the NGBs is proportional to the Fermion mass times the SSB vev.

• This was all first observed in real-world physics, which guided theorists to the understanding mentioned above. First came $SU(2)_I$ isospin, with $\begin{pmatrix}p\\n\end{pmatrix}$ an $I=\frac{1}{2}$ doublet and $\pi^0,\pi^\pm$ an $I=1$ triplet. We now understand this as coming from the approximate $SU(2)_{L+R}$ that rotates the lightest quark flavors: $\begin{pmatrix}u\\d\end{pmatrix}$ form an $I=\frac{1}{2}$ doublet. This approximate symmetry is unbroken by $SU(3)_C$, and the $\pi^0,\pi^\pm$ are the NGBs for $SU(2)_L\times SU(2)_R\to SU(2)_D$. The energy scale for this breaking is matched by fitting to the observed pion interactions, which gives $f_\pi\approx190MeV$ (and $\Lambda_{QCD}\approx300MeV$). The small quark masses (really, the Yukawa couplings to the Higgs), and also the electromagnetic and weak forces mildly explicitly break this symmetry, e.g. $m_u\approx2MeV$, $m_d\approx4.8MeV$. The masses of the three lightest pions are $m_{\pi^0}=135MeV$ and $m_{\pi^\pm}=140MeV$. The π^+ contains $u\bar{d}$

quarks and has a lifetime of $\sim 10^{-8}s$ $(\pi^+ \to \bar{\mu} + \nu_\mu)$, and π^0 contains $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$ quarks with a lifetime of $10^{-16}s$ $(\pi^0 \to 2\gamma)$.

This was then extended to the "eighfold way". The $N_f = 3$ light quark flavors (u, d, s) $(m_s \approx 92MeV)$ have an approximate $SU(3)_L \times SU(3)_R$ global symmetry, which is respected by the $su(3)_c$ strong force but broken explicitly by the non-zero masses, and by $su(2)_W \times u(1)_Y$ and by $u(1)_{E\&M}$. This approximate global symmetry is spontaneously broken by $\langle \psi\tilde{\psi} \rangle \neq 0$, leading to approximate NGBs in the adjoint of the diagonal $SU(3)_F$. There are indeed 8 light pions which fit this perfectly:

$$\begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{-\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -2\frac{\eta^0}{\sqrt{6}} \end{pmatrix} \in \mathbf{8}.$$

The baryons, including the proton and the neutron (which together are in the fundamental of $SU(2)_I \supset SU(3)_F$) and others, also form $SU(3)_F$ representations: the spin $j = \frac{1}{2}$ baryons are in the $\mathbf{8}$, the $j = 3/2$ baryons are in the $\mathbf{10}$ of $SU(3)_F$. Gell-Mann and independently Ne'eman used broken approximate $SU(3)_F$ to predict the existence, and the mass, of a baryon that is now understood to be made up from three strange quarks in the $\mathbf{10}$ of $SU(3)_F$ and with spin $j = 3/2$. Note that this is completely symmetric in the $SU(3)_F$ labels and in the spin, and this fits with Fermi statistics because it is completely antisymmetric in $su(3)_c$ to get something color neutral.

• Consider general $SU(N_c)$ with N_f flavors, and the $G/H \cong SU(N_f)$ low-energy non-linear sigma model. This space has non-trivial topology: it contains a S^3 so $\pi_3(G/H) = \mathbf{Z}$ for $N_f \geq 2$. For $N_f \geq 3$ it also contains a S^5, so $\pi_5(G/H) = \mathbf{Z}$ for $N_f \geq 3$; this S^5 (more precisely, the non-trivial H^5 cohomology of closed mod exact 5-forms) plays a role in giving what is known as the Wess-Zumino-Witten interaction of the low-energy theory, which is needed for several reasons, some of which will be mentioned soon. The S^3 means that there can be solitons associated with the $\pi_3(S^3) = \mathbf{Z}$ winding number. Such solitons have $\pi^a(t, \vec{x})$ giving a winding map from the S^3 of space $\vec{x}$ (including the point at infinity) to the S^3 where the π^a take values. This is similar to what we discussed with instantons, but there it was interpreted as a tunneling configuration in Euclidean spacetime, mapping S^3_∞ into the gauge fields. Here the configuration is instead a particle in Minkowski spacetime, because the time coordinate of $\pi^a(t, \vec{x})$ is not Wick rotated or winding. Actually, the LEEFT with just $\mathcal{L}_2$ does not admit stable solitons: because it is scale invariant, there is nothing to set the size of the solitons and they can reduce their energy to zero

– this property is called Derrick's theorem. The simplest way to avoid this is called the Skyrme model and the solitons are called Skyrmions: one adds higher derivative terms $\Delta\mathcal{L} = \mathcal{L}_4 = \frac{1}{32g^2}\mathrm{Tr}([U^\dagger\partial_\mu U,^\dagger \partial_\nu U)^2$ and then the solitons have a BPS bound on the energy of the soliton, $E \geq (6\pi^2 f_\pi/g)|W|$ where $|W|$ is the $\pi_3(S^3)$ winding of the soliton.

The original microscopic theory had the $U(1)_B$ global symmetry that acts on baryons, which are formed from ψ^{N_c} with the color indices contracted with $\epsilon_{c_1\ldots c_{N_c}}$, and anti-baryons are $\tilde{\psi}^{N_c}$. The extreme LEEFT does not contain the baryons, so it seems that $U(1)_B$ is trivial in the LEEFT. But now there is a $U(1)_W$ global symmetry associated with winding number. It turns out that W can be identified with baryon number B. It is a highly non-trivial and nice check that the solitons indeed have the right quantum numbers to match with the baryons. For example, since the baryon is made from N_c spin $\frac{1}{2}$ Fermions, contracted with an epsilon tensor in the gauge indices, it must be symmetric in the remaining $SU(N_f)_D$ and spin $SU(2)_D$, with $(-1)^F = (-1)^{N_c}$. The global flavor quantum numbers rely on properly quantizing the collective coordinates, including for the Fermions. It also relies heavily on the WZW term.

The chiral Lagrangian has both parity $P_0 : \vec{x} \to -\vec{x}$ symmetry and $U \to U^\dagger$ symmetry, i.e. $(-1)^{N_\pi}$. The correct parity of the UV QCD theory is $P = P_0(-1)^{N_\pi}$. Sometimes the LEEFT has an extra symmetry, called accidental, that is not there in the underlying theory. This is not such a case. The apparent extra symmetry is an artifact of missing an interaction term, the WZW term, which separately breaks P_0 and $(-1)^{N_\pi}$. It cannot be written as an integral over a 4d Lagrangian, but instead requires going to a 5d space Y with our spacetime as its boundary, $X_{4d} = \partial Y$. The WZW term is $S_{WZW} = k\int_Y \omega_5$, where ω_5 is the volume 5-form on G/H, $\omega \sim \mathrm{Tr}(U^\dagger dU)^{\wedge 5}$. The 5d action is not a total derivative, so the answer for different Y's generally differs. The difference is $\int_{Y-Y'\cong S^5}\omega_5 = 2\pi\pi_5(S^4) in 2\pi\mathbf{Z}$ since ω is normalized to give the S^5 winding number. Then $e^{ik\int_Y \omega_5}$ is invariant under this ambiguity if $k \in Z$. Witten showed that everything works correctly, including anomaly matching and getting the right quantum numbers of the baryon from the skyrmionic soliton, if $k = N_c$.

• If $U(1)_A$ were a symmetry, there would have to be a 9th pseudoscalar (since it is P odd) meson; the candidate observed particle is called the η', but it is too massive to be considered an approximate NGB. Estimates for its mass based on the quark masses suggested $m_{\eta',wrong} \approx 355MeV$ whereas $m_{\eta',actual} \approx 958MeV$. The resolution is that $U(1)_A$ is not a symmetry, because it has a quantum anomaly, and this gives the η' a large mass compared to the light pions.

Before discussing anomalies in more detail, let's mention another related puzzle: the $\pi^0 \to \gamma\gamma$ decay lifetime. The decay proceeds by an interaction term in the LEEFT that is familiar from our discussion of instantons, axions, and a previous HW exercise: $\mathcal{L} \supset g\pi^0 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$. This is consistent with the fact that π^0 is a parity-odd pseudoscalar, but it is inconsistent with π^0 being a true NGB: it is not really a derivative coupling (well, we can integrate by parts to write it as $\partial_\mu \pi^0 K^\mu$ but K^μ is not gauge invariant). The coupling leads to a decay rate $\Gamma(\pi^0 \to 2\gamma) = m_\pi^3 g^2/\pi$. A naive estimate of g is that it is generated by a loop, so $g \sim \alpha/2\pi F_\pi$. A seemingly less naive estimate is that this decay violates the chiral symmetry, which suggests that an extra factor of $m_u + m_d \sim m_\pi^2$ is needed, which should be m_π^2/m_N^2. The original estimate leads to $\Gamma_{naive} \sim 2 \times 10^{16} s^{-1}$ and the "improved" version leads to $\Gamma_{less(?)naive} \sim 4 \times 10^{16} s^{-1}$. Observation gives $\Gamma_{obs} \sim 10^{16} s^{-1}$, so approximate $U(1)_A$ symmetry again does not fit with observation. This was connected to the triangle diagram anomaly in 1969 by Bell and Jackiw, and Adler.