215c, 6/1/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 10 reading: Tong chapter 3 on anomalies.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Anomalies arise only at one-loop, from a $\frac{1}{2}(d+2)$-gon diagram, which contributes to a $\frac{1}{2}(d+2)$ point function of currents – in 4d, it is 3-point function of currents, and arises from a triangle diagram. The anomaly is topological and is independent of the coupling when the gauge fields are correctly normalized. It is also independent of mass parameters, which again shows that it can only arise from massless fields – for massive fields, the mass can be taken to infinity and the field then decouples.

• Writing $Z[A] = \int [d\psi] \exp(\frac{i}{\hbar}(S[\psi] + A_\mu J^\mu))$, an anomaly means that $Z[A]$ is not gauge invariant, $Z[A^\lambda] = Z[A] \exp(-2\pi i \int_X \alpha(\lambda, A))$, where λ is the gauge transformation parameter and α involves an epsilon tensor, which is why it has the i in the exponent in either Minkowski or Euclidean space, as in our discussion of the theta term in Euclidean space. The epsilon tensor allows the indices to be contracted without using the metric, and the result is topological. The anomalous variation t implies that the associated Noether current is not conserved or covariantly conserved.

• The anomaly can be understood as the statement that the path integral measure $[d\psi]$ does not respect the symmetry.

• For a gauge symmetry, we still need to integrate $\int [dA]/(\sim) Z[A]$, where / means modulo gauge transformations. The functional integral is not well defined (e.g. it integrates to give zero from adding the oscillating phases) if there is a gauge anomaly. Theories with gauge anomalies are sick, so an anomaly involving three gauge fields signals an inconsistency in the theory. This restricts the allowed matter content, and this can work out non-trivially for chiral (non-vector-like) matter content, as in the Standard Model. For vector-like matter content, (i.e. admitting mass terms for all fields compatible with the symmetries) it is automatically true, since anomalies cannot depend on m and cannot get contributions as $m \to \infty$. So $\mathrm{Tr} G_i G_j G_k = 0$ where G_i is any gauge symmetry.

• The original example of an anomaly is $\mathrm{Tr} FGG$ where F is a global flavor symmetry, and G is a gauge symmetry. Such anomalies never spoil gauge invariance – we can add counter-terms if need be to restore any apparent breaking of gauge symmetry. But they violate the flavor symmetry F. Since F appears linearly, such anomalies are proportional to $\mathrm{Tr} T_F$ and thus vanish for any non-Abelian flavor symmetry. They can only be non-zero for a $U(1)_F$ flavor symmetry. For example, for $SU(N_c)$ with N_f Dirac Fermions, we saw that there is a classical $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$ flavor symmetry. We immediately know that $SU(N_f)_{L+R}$ and $U(1)_V$ cannot be anomalous, because they are vector-like. Likewise $SU(N_f)_L$ and $SU(N_f)_R$ cannot have an anomaly because their generators are traceless. So only $U(1)_A$ can (and does) have a non-zero anomaly. The $\mathrm{Tr}U(1)_F G^2$ anomaly is equivalent to the statement that an instanton configuration has Fermion zero modes. In this sense, the anomaly was first found by the mathematicians Atiyah and Singer in their 1963 proof of their index theorem, and independently found by Adler and Bell and Jackiw in 1969 (the relation between the two was not initially apparent, and only understood later). The interpretation of the anomaly is that the $U(1)_F$ classical symmetry is explicitly violated non-perturbatively, by instantons (a discrete remnant can remain). The tell-tale sign of the violation is a one-loop diagram, and the fact that the current cannot be modified to cancel the effect is only seen non-perturbatively, for the same reason that the $\vec{E} \cdot \vec{B}$ term that enters in the theta angle looks like a trivial total derivative, but has non-trivial winding in the instanton configuration. For example, $U(1)_A$ for $SU(N_c)$ with N_f flavors has an anomaly equal to $2N_f$ times the instanton density, corresponding to the fact that an instanton has a Fermi zero mode for each ψ_α and $\tilde{\psi}_\alpha$ flavor.

• 't Hooft pointed out in the Cargese Summer institute in 1979 that anomalies involving three global symmetries are also interesting: they are an obstruction to gauging the global symmetry. E.g. in the $SU(N_c)$ example $\mathrm{Tr}SU(N_f)_L^3 = -\mathrm{Tr}SU(N_f)_R^3 = N_c$. 't Hooft argued essentially that such anomalies must be constant on RG flows, so if they are non-zero then the theory cannot be gapped in the IR , and the IR massless spectrum must match the 't Hooft anomalies of the UV theory.

• Let's sketch how the non-zero anomaly is computed in a characteristic example. All examples are similar, modulo combinatoric differences and group theory factors from the matter charge assignments, which are multiplicative factors. We will discuss two examples in parallel: the $\mathrm{Tr}U(1)_A U(1)_V$ axial anomaly in 2d, and the $\mathrm{Tr}U(1)_A U(1)_V^2$ axial anomaly in 4d. In 4d, as a simple example, consider a massless Dirac Fermion, Ψ, which can be written as two chiral Weyl Fermions, e.g. ψ_α and $\tilde{\psi}_\alpha$. Classically we can rotate them with separate $U(1)_L \times U(1)_R$ phases, with $U(1)_V = U(1)_{L-R}$ and $U(1)_A = U(1)_L + U(1)_R$; the corresponding currents are $J_{V,\mu} = \bar{\Psi}\gamma^\mu\Psi$ and $J_{A,\mu} = \bar{\Psi}\gamma^\mu\gamma^5\Psi$. The anomaly is the statement that it is impossible in the quantum theory to preserve both symmetries; we can choose to preserve $U(1)_V$ and then sacrifice $U(1)_A$. Couple $U(1)_V$ to a gauge field A_μ (either a dynamical one, if the symmetry is gauged, or a background source if it is not) and couple $U(1)_A$ to a background gauge field $A_{5,\mu}$. In 2d, the anomaly is associate with $\partial_\mu J_A^\mu \neq 0$ in the two-point function $\langle T J_A^\mu(x) J_V^\nu(0)\rangle$ and in 4d with $\partial_\mu J_A^\mu \neq 0$ in the three-point function $\langle T J_A^\mu(x) J_V^\nu(y) J_V^\rho(0)\rangle$. With non-zero background A_μ for $U(1)_V$, the result can be stated as $\partial_\mu J_{A,2d}^\mu \sim \frac{1}{2\pi}\epsilon^{\rho\sigma}F_{\rho\sigma}$ (i.e. proportional to $*c_1(F)$) and $\partial_\mu J_{A,4d}^\mu \sim \frac{1}{16\pi^2}\epsilon^{\rho\sigma\kappa\lambda}F_{\rho\sigma}F_{\kappa\lambda} \sim \vec{E}\cdot\vec{B}$ (i.e. proportional to $*c_2(F)$).

Consider the 2d case first, with $\mathcal{L}_{2d} = i\psi_R(D_t + D_z)\psi_R + i\psi_L(D_t - D_z)\psi_L$, with $D_\mu = \partial_\mu - ieA_\mu$. The $\psi_{R,L}$ Fermions have $p_z = \pm E$, and groundstate for $A_\mu = 0$ consists of the filled Dirac sea for $E < 0$, and unoccupied levels for $E > 0$. A constant external electric field $\vec{E} = \mathcal{E}\hat{z}$ can be applied, $A_z = -\mathcal{E}t$ to get $\Delta E_{R,L} = \pm e\mathcal{E}t$: the electric field shifts the Dirac sea up for the right movers and down for the left movers – the density of states $dp_z/2\pi$ shifts as $\frac{d}{dt}\rho_{R,L} = \pm\frac{e\mathcal{E}}{2\pi}$. The vector density $\rho_R + \rho_L$ is preserved, but the axial density $\rho_A = \rho_R - \rho_L$ has $\dot{\rho}_A = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$.

Now consider the 4d case with both an electric and magnetic field. An external magnetic field $\vec{B} = B\hat{z}$ leads to Landau levels associated with quantizing the circular orbits, so $H \to p_z^2 + (2n+1)eB - 2eBS_z$; the density of states in the n-th Landau level is $g_n = eB/2\pi$ (as seen by matching to $d^3p/(2\pi)^3$ in the $B \to 0$ limit). Effectively get a $1+1$ dimensional theory of Fermions with mass $m^2 = (2n+1)eB - 2eBS_z$, and this relates the anomaly in 4d to the 2d chiral anomaly. For $n = 0$ and $S_z = \frac{1}{2}$, get $m = 0$ for the 2d Fermion. For a right (left) handed chiral Fermion in 4d, this gives a 2d right (left) mover along the $\hat{z}$ axis. To get the 4d anomaly, multiply the 2d anomaly by the density of states g_n in the (x, y) plane to get $\dot{\rho}_A = \frac{e^2}{2\pi^2}\vec{E}\cdot\vec{B} = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$.

• The e^2 in the anomaly equation is a remnant of our normalization choice for the gauge fields, as was highlighted earlier. For perturbative calculations, it is best to use A_μ^{pert}, which is normalized such that $D_\mu = \partial_\mu - ieA_\mu^{pert}$ and interaction vertex then comes with a factor of the coupling e (or g in the non-Abelian case). For conceptual clarity, it is best to use $A_\mu^{best} = eA_\mu^{pert}$, which couples to the currents without any factors of e, and D_μ is independent of the coupling; then the only appearance in the coupling is as a $1/e^2$ in front of the kinetic terms (i.e. the gauge field propagator gets a factor of e^2). In this normalization, electric charges (via Gauss' law) are quantized to be integers, rather than integer multiples of the coupling constant e. In this normalization, $\dot{\rho}_A = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ and for comparison we had $S_\theta = \int \frac{\theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ with $\theta \sim \theta + 2\pi$ multiplying the instanton density. The factor of 2 is because we considered $U(1)_A$ gives charge $+1$ to both ψ_α and $\tilde{\psi}_\alpha$ – if we gave charge to only ψ or $\tilde{\psi}$, $\dot{\rho}$ would be the instanton density on the nose.

• Let's consider the anomaly from the perspective of the path integral $Z[A] = \int [d\psi][d\bar\psi]\exp(-\int d^4x\bar\psi(i\not{D})\psi) = \prod_n \lambda_n = \det(i\not{D})$, where $i\not{D}\psi_n = \lambda_n\psi_n$. Here $D_\mu = \partial_\mu - ieA^\mu$, with A^μ the $U(1)_V$ gauge field, and $U(1)_A$ acts as $\psi \to e^{\epsilon\gamma^5}\psi$ and $\bar\psi \equiv \psi^\dagger\gamma^0 \to \bar\psi e^{i\epsilon\gamma^5}$, so $\det(i\not{D}) \to \det(e^{i\epsilon\gamma^5} i\not{D} e^{i\epsilon\gamma^5})$ and if not for the fact that the det involves an infinite product we could use $e^{i\epsilon\gamma^5}\gamma^\mu e^{i\epsilon\gamma^5} = \gamma^\mu$. The measure $[d\psi][d\bar\psi] \to \prod_n d\psi_n d\bar\psi_n$ and the Jacobian is infinite and can be regulated by introducing a factor of $e^{-\lambda_n^2/\Lambda^2} = e^{-(i\not{D})^2/\Lambda^2}$ and taking $\Lambda \to \infty$. The Jacobian is then $e^{-2i\epsilon \mathrm{Tr}(\gamma^5 e^{-(i\not{D})^2/\Lambda^2})}$ where the $-$ sign is because it's a Jacobian for Grassmann integration. Then use $\mathrm{Tr}\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4i\epsilon^{\mu\nu\rho\sigma}$ (the i is there in Euclidean space) from expanding the exponent, with $\not{D}^2 = D_\mu D^\mu - \frac{1}{2}ie\gamma^\mu\gamma^\nu F_{\mu\nu}$. Get $\mathrm{Tr}\gamma^5 e^{\not{D}^2/\Lambda^2} = i\int \frac{d^4k_E}{(2\pi)^4} e^{-k_E^2/\Lambda^2}\mathrm{Tr}(\gamma^5 \frac{1}{2}e^2\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\frac{1}{\Lambda^4} + \ldots) = \frac{e^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$. So $[d\psi][d\bar\psi] \to [d\psi][d\bar\psi]\exp(-\frac{ie^2}{16\pi^2}\int d^4x\epsilon(x)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma})$.

This has the same effect as shifting the θ angle, $\Delta\theta = -2\epsilon$. The 2 is because we considered a theory with two Fermions, ψ_α and $\tilde\psi_\alpha$, both with $U(1)_A$ charge $+1$.

• Yet another way to understand the anomaly is by point-splitting the operator $j_5^\mu = \lim_{\epsilon\to 0}\bar\psi(x+\frac{1}{2}\epsilon)\gamma^\mu\gamma^5\exp(-ie\int_{x-\frac{1}{2}\epsilon}^{x+\frac{1}{2}\epsilon} A_\nu(y)dy^\nu)\psi(x-\frac{1}{2}\epsilon)$, where the charged operators are connected by an open Wilson line. Then $\langle\partial_\mu j_5^\mu\rangle$ has a term where the ∂_μ hits $\bar\psi$, one where it hits ψ, and one where it hits the A_ν in the exponent. Using the EOM, this leads to $\langle\partial_\mu j_5^\mu\rangle = \lim_{\epsilon\to 0}\langle\bar\psi(x+\frac{1}{2}\epsilon)\exp(-ie\gamma^\mu\epsilon^\nu F_{\mu\nu}(x))\gamma^5\psi(x-\frac{1}{2}\epsilon)\rangle$. The anomaly follows upon taking the contraction of $\bar\psi(x+\frac{1}{2}\epsilon)$ and $\psi(x-\frac{1}{2}\epsilon)$ in the presence of the background gauge field A_μ. In 2d, the term in the contraction that is independent of A_μ gives the anomaly. In 4d, this term in the contraction is $-i\gamma^\alpha\epsilon_\alpha/2\pi^2\epsilon^4$, and multiplying by γ^5 gives 0 since $\mathrm{Tr}\gamma^\mu\gamma^5 = 0$. Expanding to next order in the background gauge field gives

$$\langle\bar\psi(x+\epsilon/2)\gamma^\mu\gamma^5\psi(x-\epsilon/2)\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}e^{ik\epsilon}e^{-ipx}\mathrm{Tr}[\gamma^\mu\gamma^5\frac{i}{(\not{p}+\not{k})}(-ie\not{A}(p))\frac{i}{\not{k}}].$$

This equals $i\frac{e\epsilon_\kappa}{4\pi^2\epsilon^2}\epsilon^{\mu\rho\sigma\kappa}F_{\rho\sigma}(x)$ in the $\epsilon\to 0$. Plugging into the above, and symmetrizing in the $\epsilon\to 0$ limit (eliminating the singular term in the operator product expansion when we define the composite operator) reproduces the anomaly.