215c, 4/8/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 2 reading: Tong chapter 1, and start chapter 2.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Continue from last time with the θ term.

$$S_{\theta,u(1)} = \int d^4x \frac{\theta}{4\pi^2\hbar c}\vec{E}\cdot\vec{B} = \int d^4x \frac{\theta}{32\pi^2\hbar c}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \hbar\int \theta c_1(F)\wedge c_1(F).$$

The normalization is such that $S_\theta = \hbar\theta k$ with $k\in\mathbf{Z}$ an integer (called the instanton number). Then $e^{iS/\hbar}$ in the path integral is periodic in $\theta\cong\theta+2\pi$. There are interesting variants, with different periodicities (e.g. 4π) on certain spacetime manifolds, and a discussion of such cases could make for a good final presentation topic.

• If θ is replaced with a dynamical scalar field, the field is called the *axion* a (we had a recent colloquium by Frank Wilczek about this). The axion's field target space is a circle, $a\cong a+2\pi$. Let's replace $\theta\to\theta(t,\vec{x})$, which could either be the dynamical axion or an external background source for the operator $\tilde{F}^{\mu\nu}F_{\mu\nu}$. The classical EOM for A_μ become:

$$\nabla\cdot\vec{E} = j_e^0 - \frac{\alpha c}{\pi}\nabla\theta\cdot\vec{B}, \qquad \alpha\equiv e^2/4\pi\hbar c,$$

$$-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \nabla\times\vec{B} = \vec{j}_e + \frac{\alpha}{\pi c}(\dot{\theta}\vec{B} + \nabla\theta\times\vec{E}),$$

where j_e^μ is the electric current associated with matter fields. The fact that θ enters the EOM only via derivative terms, so it drops out of the EOM if θ is a constant, fits with the fact that θ is the coefficient of a total derivative term. The fact that it is the total derivative of a gauge non-invariant does not affect the fact that it drops out of the EOM; this is a general aspect of topological terms. For example, there are intergalactic magnetic fields which, with a varying θ, can source electric fields.

In absence of electric and magnetic sources, we set $j_e^\mu=0$, and $j_m^\mu=0$, and the other two Maxwell equations are unchanged, $\nabla\cdot\vec{B}=0$ and $\partial_t\vec{B}+\nabla\times\vec{E}=0$. If we define $\vec{D}=\epsilon_0(\vec{E}+\frac{\alpha c\theta}{\pi}\vec{B})$ and $\vec{H}=\mu_0^{-1}(\vec{B}-\frac{\alpha\theta}{\pi c}\vec{E})$, then $\nabla\cdot\vec{D}=0$ and $\nabla\times\vec{H}-\partial_t\vec{D}=0$. At an interface where θ changes, there are associated surface charges and currents. If there are no other external charges and currents, these follow from $\hat{n}\cdot\Delta\vec{D}=0$ and $\hat{n}\times\Delta\vec{H}=0$. E.g. there are topological insulator materials which effectively have $\theta=\pi$. Consider an interface, e.g. $\theta=\pi\Theta(-z)$. Then if an external source makes $\vec{B}$, get surface electric charge density $\sigma=\alpha cB\hat{n}$. If an external source makes $\vec{E}$, get surface current density $\vec{K}=\alpha\epsilon_0 c\vec{E}\times\hat{n}$. This is a Hall conductivity $\sigma=\frac{1}{2}\cdot\frac{e^2}{2\pi\hbar}$.

• Dyons are objects with both electric charge q_e and magnetic charge q_m. The charge pair is written as (q_e, q_m), so a basic electric charge is $(1, 0)$ and a basic magnetic charge is $(0, 1)$. The generalization of Dirac's quantization argument is to consider an object of charges $(q_{1,e}, q_{1,m})$ in the background of an object with charges $(q_{2,e}, q_{2,m})$ and the resulting Dirac-Zwanziger quantization condition is $q_{1,e}q_{2,m} - q_{2,e}q_{1,m} \in 2\pi\hbar Z$.

• The Witten effect: θ gives electric charge $\sim \theta$ to magnetic monopoles. Suppose we take the minimum electric charge to be $q_e = 1$, and the minimum magnetic charge is then $q_m = 2\pi\hbar$. The minimum magnetic charge monopole becomes a dyon with electric charge $q_e = \frac{\theta}{2\pi}$. This can be understood from $\partial_\mu\theta$ terms in the above EOM for $\vec{E}$ and $\vec{B}$. If θ is rotated from 0 to 2π, the original monopole becomes a bound state of a monopole and an object of electric charge 1. There is a generalization for dyons, and it is compatible with the Dirac-Zwanziger quantization condition (θ cancels).

• The gauge group of the Standard Model is $su(3)_C \times su(2)_W \times u(1)_Y$ (there are some nice fine points about discrete symmetries that I might return to later, but ignore here). The $u(1)_Y$ part is similar to the above, though with a crucial difference that the left and right-handed chiral Fermions have different q charges. The $su(3)_C$ and $su(2)_W$ gauge symmetries are somewhat similar, but there are several crucial differences. First, these symmetries are non-Abelian (recall the non-Abelian nature of $su(2)$ as seen by rotating a book along two axes in different orders giving different final orientations). Also, $su(2)_W$ is chiral: it only acts on left-handed Fermions; this is why the weak interactions violate P. Also, $su(2)_W$ and $u(1)_Y$ are broken to a $u(1)_{EM}$ subgroup by the vacuum expectation value of a Bose condensate (the Higgs field), which is why the $W^{\pm}_\mu$ and Z^μ gauge fields are massive. I intend to discuss these and related topics further in this class.

• Let's start by discussing the analogs of D_μ and $F_{\mu\nu}$ in a gauge theory with non-Abelian group G. To be concrete, I will sometimes take $G = SU(2)$, but most of the discussion for now will be general. I will also sometimes use the notation of writing the gauge group, in lower case letters, e.g. $su(2)$, to distinguish a local gauge symmetry vs global symmetry. We can consider a pure Yang-Mills theory, which means only gauge fields and no matter representations, or we can include matter fields. The matter fields could be scalars (e.g. the Higgs in the SM) ϕ or Fermions ψ (as in QED, these are Dirac if $m \neq 0$ or, for $m = 0$, we can have left and / or right-handed chiral fermions – more on this shortly). The matter fields are in representations of the gauge group, and the most-discussed cases are the fundamental, anti-fundamental, and adjoint representations;

for $su(2)$ the fundamental and anti-fundamental are $\mathbf{2} \cong \bar{\mathbf{2}}$ (like $j = \frac{1}{2}$ for the rotation group), and the $\mathbf{3}$ adjoint representation (like $j = 1$ for rotations).

Consider e.g. $su(N)$ with ψ^c_α in the fundamental, where $\alpha = 1, 2$ is a spinor index and $c = 1, \dots N$ is the $su(N)$ color index; the Lorentz index α goes for the ride and will usually be suppressed. The $su(N)$ gauge symmetry takes $\psi^c \to U^c{}_d(x)\psi^d$ where we sum the repeated color index. Here $U(x)$ is an element of the $SU(N)$ group manifold (e.g. for $SU(2)$ it's $\cong S^3$): $U^\dagger U = \mathbf{1}$ and $\det U = 1$, and we can thus write it as the exponential of Hermitian, traceless $N \times N$ matrices; there are $N^2 - 1$ of these and for the rotation group these are the 3 generators J^a. An anti-fundamental can be written as $\tilde{\psi}_c$ which transforms as $\tilde{\psi}_c \to U^{*d}{}_c\tilde{\psi}_d$ and $U^\dagger U$ means that δ^c_d is invariant and $\det U = 1$ means that $\epsilon_{c_1 \dots c_N}$ and $\epsilon^{c_1 \cdots c_N}$ are invariant. For $SU(2)$, the fundamental and anti-fundamental are related by $\tilde{\psi}_c = \epsilon_{cd}\psi^d$.

• We now want to write a covariant derivative, such that $\psi \to U\psi$ takes $D_\mu\psi \to D^U_\mu U\psi = UD_\mu\psi$, so $D^U_\mu = UD_\mu U^{-1}$. Let's write it as $D_\mu = \partial_\mu - iA_\mu$ (this changes the sign of A_μ vs my convention for $u(1)$, to agree with the notation for the non-Abelian case in other references), with $A_\mu = A^a_\mu T^a$. Note that, when we write $D_\mu = \partial_\mu - iA_\mu$, it should be understood that $A_\mu = A^a_\mu T^a_R$, with R the representation of the field that it acts on. For example, if acting on something that is invariant, i.e. the trivial rep, then $T^a_R = 0$ and $D_\mu \to \partial_\mu$ (similar to how in GR the ∇_μ notation has implicit the connection, which depend on what it acts on, with e.g. $\nabla_\mu \to \partial_\mu$ if acting on a scalar). For $u(1)$, the generator $T^a_R \to q$, the electric charge of the operator on which it acts. For $su(2)$, if acting on the rep labeled by I (analogous to j for the rotation group), then e.g. $D_\mu = \partial_\mu \mathbf{1}_{2I+1} + iA^a_\mu T^a_{2I+1}$, where from now on I will not explicitly write the $\mathbf{1}$ or the R reminder that D_μ depends on the representation of the field on which it acts.

We want $D^U_\mu(U\psi) = UD_\mu\psi$, i.e. $\partial_\mu U - iA^U_\mu U = -iUA_\mu$. So $A^U_\mu = iU\partial_\mu U^{-1} + UA_\mu U^{-1}$. As a check, for $U(1)$ with $U = e^{i\alpha}$ this gives $A^U_\mu = \partial_\mu\alpha + A_\mu$. More generally, taking $U = \exp(i\alpha)$, with $\alpha = \alpha^a T^a$ and expand for an infinitesimal α^a, get $\delta A_\mu = D_\mu\alpha = \partial_\mu\alpha + [iA_\mu, \alpha]$. Writing $D^a_\mu = \partial_\mu - iA^a_\mu$, $(D_\mu\alpha)^a = \partial_\mu\alpha^a - A^b_\mu\alpha^c f^{abc}$.

The gauge field strength tensor corresponds to a commutator of covariant derivatives (analogous to the Riemann curvature tensor):

$$F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = F_{\mu\nu} = F^a_{\mu\nu}T^a,$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu.$$

$F_{\mu\nu}$ is in the adjoint rep, transforming as $F_{\mu\nu} \to F^U_{\mu\nu} = UF_{\mu\nu}U^{-1}$. For $U = \exp(i\alpha^a T^a)$ and α^a infinitesimal, get $\delta F_{\mu\nu} = i[\alpha, F_{\mu\nu}]$, which is the statement that $F_{\mu\nu}$ transforms in the adjoint rep. The Lagrangian density must of course be gauge invariant, and the gauge kinetic terms come from the quadratic casimir (squaring and taking the trace): $\mathcal{L} \supset -\frac{1}{2g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$. E.g. for $G = SU(2)$, we can use a notation inspired by the rotation group, where the $j = 1$ adjoint is denoted by a 3d vector, so $\vec{F} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + \vec{A}_\mu \times \vec{A}_\nu$, and the gauge kinetic terms are $-\frac{1}{4g^2}\vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu}$.