215c, 4/20/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 4 reading: Tong chapter 2.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Recall from last time that in $A_0 = 0$ gauge $S_\theta = \theta \int dt \partial_0 W[A]$ where $W[\vec{A}] = \frac{1}{8\pi^2}\int d^3x \epsilon^{ijk}\mathrm{Tr}(A_i\partial_j A_k - \frac{2i}{3}A_iA_jA_k)$ is gauge invariant under $\vec{A} \to U(\vec{A} + i\vec{\nabla})U^{-1}$ up to shifts by $n[U] \in \mathbf{Z}$ which measures $\pi_3(G)$ from $R^3|_{t=const} + \{\cdot_\infty\} \cong S^3 \to U$. In $A_0 = 0$ gauge, $\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}F^{\mu\nu}F_{\mu\nu} + \frac{\theta}{16\pi^2}\mathrm{Tr}{}^*F^{\mu\nu}F_{\mu\nu}$ gives

$$\mathcal{L} = \frac{1}{g^2}\mathrm{Tr}(\dot{\vec{A}}^2 - \vec{B}^2) + \frac{\theta}{4\pi}\mathrm{Tr}\dot{\vec{A}} \cdot \vec{B},$$

so $\vec{\Pi} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{A}}} = g^{-2}\vec{E} + \frac{\theta}{8\pi^2}\vec{B}$ and $\mathcal{H} = g^{-2}\mathrm{Tr}(\vec{E}^2 + \vec{B}^2) = g^2\mathrm{Tr}(\vec{\Pi} - \frac{\theta}{8\pi^2}\vec{B})^2 + g^{-2}\mathrm{Tr}\vec{B}^2$. Even though we take $A_0 = 0$, we still need to impose the EOM for A_0, which gives a non-Abelian version of Gauss' law, $\mathcal{D}_iE_i = 0$. As usual In QFT we do not impose EOMs as operator equations, but just that it holds for on-shell physical states, so $\mathcal{D}_iE_i|\mathrm{physical}\rangle = 0$. The Hilbert space includes non-gauge invariant states, and Gauss' law is equivalent to the condition that the physical states should be charge neutral.

• The effect of the θ term is similar to the θ term that we saw for QM with magnetic flux inside. There were two ways to see it: we could either take $H = \frac{1}{2mR^2}(-i\partial_\phi + \frac{\theta}{2\pi})^2$ and impose $\psi(\phi + 2\pi) = \psi(\phi)$, or we could try to eliminate θ by something that is roughly just a gauge transformation $\psi' = e^{i\theta\phi/2\pi}$ and then $H = \frac{1}{2mR^2}(-i\partial_\phi)^2$, but then $\psi(\phi + 2\pi) = e^{i\theta}\psi(\phi)$ has twisted boundary conditions. This shows that $\theta \sim \theta + 2\pi$. Likewise, there are two equivalent ways to see the θ parameter in gauge theories. Consider the functional $\Psi(\vec{A}) = \langle \vec{A}|\Psi\rangle$. We can then take $\vec{\Pi} \to -i\frac{\delta}{\delta\vec{A}}$ and write a SE with the $\mathcal{H}$ above, which involves θ. Alternatively, we can consider $\Psi'(\vec{A}) = e^{i\theta W[\vec{A}]}\Psi(\vec{A})$, which eliminates the θ term in the SE, but Ψ' satisfies twisted boundary conditions: $\Psi'(A) \to e^{in\theta}\Psi(A)$ under a large gauge transformation with $\pi_3(G)$ winding n. This shows that $\theta \sim \theta + 2\pi$. Aside: there are some interesting fine points that are being glossed over here. For example, there are different version of $su(N)$ Yang-Mills that differ in global observables – Wilson and 't Hooft lines – e.g. $su(N)$ which has $\theta \sim \theta + 2\pi$ vs $su(N)/Z_N$ which has $\theta \sim \theta + 2\pi N$; see Tong's notes for a nice discussion and details. In terms of the Standard Model, the group is $su(3)_C \times su(2)_W \times u(1)_Y/\Gamma$ were Γ could be 1, Z_2, Z_3, or Z_6; again, see Tong for details. These could be a good topics for your final presentation.

• The θ term multiples the instanton density, which is best understood in Euclidean spacetime. We discussed the Euclidean Wick rotation in 215a. Recall that the Feynman propagator goes above the ω_k pole and below the $-\omega_k$ pole and that we can thus rotate $k_0 \to e^{+i\alpha} k_0$ with $0 \le \alpha \le \pi/2$ to make the integral go up the imaginary axis, so $k_0 \to +ik_0$, and x_0 should rotate oppositely (to keep the FT well defined), $x_0 \to -ix_0$. The Euclidean action is then $-i$ times the continuation of the Minkowski action, e.g. starting with $S = \int dt(\frac{1}{2}\dot{\phi}^2 - V(\phi))$ we rotate $t \to -it$ to get $S \to i\int dt(\frac{1}{2}\dot{\phi}^2 + V(\phi))$ and get $S_E = \int dt(\frac{1}{2}\dot{\phi}^2 + V(\phi))$ which differs from the action by $V \to -V$ (the Lagrangian in Euclidean space is the original Hamiltonian). As we will discuss more, the classical trajectories in Euclidean space are thus related to tunneling. In the path integral, $e^{iS/\hbar} \to e^{-S_E/\hbar}$ and this illustrates that the $Z = \mathrm{Tr} e^{-\beta H}$ comes from Euclidean time with periodicity β. But the S_θ term behaves differently: the usual factor of i from the d^4x cancels against an i from the $\epsilon^{\mu\nu\rho\sigma}$ contraction; this reflects the fact that the term is topological. The upshot is that the S_θ term has an explicit factor of i in S_E, so it is still an oscillating contribution to the Euclidean path integral; this is indeed needed for $\theta \to \theta + 2\pi$ to still hold.

Euclidean spacetime is $\sim \mathbf{R}^4$ and we can get finite action by requiring the gauge fields to approach pure gauge at infinity. We can think of infinity as S^3_∞ and the pure gauge condition allows for non-trivial winding number $k \in \mathbf{Z}$. Here k is the instanton number and measured by the same winding number integral as n above – the different letter is because here it is associated with the Euclidean S^3_∞ with a slightly different physical interpretation. It turns out that it is inconsistent to restrict to $k = 0$ in the functional integral: we must take $[dA] \to \sum_{k=-\infty}^{\infty} [dA]_k$, where $[dA_k]$ is a sector with instanton number k. The sector with instanton number k has $\int c_2(F) = k$, and enters the path integral with a factor of $e^{ik\theta}$. When we connect back to Minkowski spacetime, the Euclidean configuration with instanton number k can be thought of as a tunneling process, between vacua at $t = \pm\infty$ with winding number $n_\pm$ with $n_+ - n_- = k$.

• In Euclidean space, the Yang-Mills action becomes $S_{YM} = +\frac{1}{2g^2}\int d^4x \mathrm{Tr} F_{\mu\nu}F^{\mu\nu}$ i.e. whereas $\mathcal{L}_M \sim \mathrm{Tr}(\vec{E}^2 - \vec{B}^2)$, the Euclidean rotation gives $\mathcal{L}_E \sim \mathcal{H} \sim \mathrm{Tr}(\vec{E}^2 + \vec{B}^2)$. In Minkowski space, $** = -1$, e.g. $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ has $\vec{E} \to \vec{B} \to -\vec{E}$, so doing it twice takes $\vec{E} \to -\vec{E}$ and $\vec{B} \to -\vec{B}$, i.e. $** F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} * F^{\mu\nu} = -F_{\mu\nu}$. In Euclidean space, $** = +1$, e.g. $*$ takes $\vec{E} \leftrightarrow \vec{B}$, and so $** F = F$. So

$$S_{YM} = \frac{1}{4g^2}\int d^4x \mathrm{Tr}(F_{\mu\nu} \mp *F_{\mu\nu})^2 \pm \frac{1}{2g^2}\int d^4x \mathrm{Tr} F_{\mu\nu} * F^{\mu\nu} \ge \frac{8\pi^2}{g^2}|k|,$$

where the inequality is saturated if $F_{\mu\nu} = \pm * F_{\mu\nu}$ and configurations with a $+$ sign are called instantons and have instanton number $k > 0$, and configurations with a $-$ sign are called anti-instantons and have $k < 0$. The instanton and anti-instanton minimize the action in their topological sector, and the action becomes $S_{YM} \to S_{inst} = \frac{8\pi^2}{g^2}|k|$. Since they minimize the action, they will automatically satisfy the EOM. Note that the EOM, $D^\mu F_{\mu\nu} = 0$ is a second order differential equation for A_μ, and this solves it via instead the first order equations $F_{\mu\nu} = \pm * F_{\mu\nu}$, roughly similar to the Hamiltonian EOM. $F_{\mu\nu} = \pm * F_{\mu\nu}$ requires $** = 1$, so it does not have an analog in Minkowski spacetime. The above is a special case of something that was studied by Bogomol'nyi Prasad Sommerfield, and the inequality is called a BPS bound. Note that $e^{-S} \to e^{-8\pi^2|k|/g^2 + i\theta k}$, which shows that the instanton contributions are non-perturbative – they do not show up in a Taylor series in g^2.

• Instantons are associated with classical solutions of the Euclidean EOM. Such solutions correspond to tunneling processes. Let's briefly illustrate this with QM. Recommended reading: Coleman's lecture on The Uses of Instantons, in Aspects of Symmetry. Consider QM with $H = \frac{1}{2}p^2 + V(x)$. If there is a potential barrier region with $V(x) > E$, the WKB approximation gives a transmission amplitude $|T(E)| \approx e^{-B}$ with $B = \int_{x_1}^{x_2} dx \sqrt{2m(V-E)}/\hbar$. This is a stationary path of the Euclidean path integral.

A classic example is QM with a double well potential: $V = \lambda(x^2 - a^2)^2$. It has a Z_2 symmetry, $x \to -x$, and the classical minima are at $x = \pm a$. This looks like spontaneous symmetry breaking of Z_2. But quantum effects – the tunneling – actually restore the Z_2 symmetry. There is a theorem that QM (aka QFT in $0+1$ dimensions) does not admit spontaneous symmetry breaking. There is a similar theorem (Coleman; Mermin-Wagner; Hohenberg)) that QFT in $d = 2$ does not admit spontaneous breaking of continuous symmetry (discrete breaking is possible in $d = 2$). If the double well barrier height is very large compared with E_0, here are approximate groundstates given by the usual SHO groundstate for each well, centered at the minimum, i.e. $|L\rangle$ and $|R\rangle$. Thanks to tunneling, the groundstate is non-degenerate, as usual and expected, and given approximately by $|+\rangle$, and the state $|-\rangle$ has slightly higher energy, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|L\rangle \pm |R\rangle)$ have parity ± 1, with $E_\pm = E_0 \mp Ke^{-B}$, where K is a calculable constant.

In the Euclidean path integral, the extremal tunneling solution comes from extrema of the classical Euclidean action. In the double well example, the classical minima at $x = \pm a$ become local maxima when $V \to -V$. There is then a classical solution that connects $|-a, -T/2\rangle$ to $\langle a, T/2|$. Taking $T \to \infty$, we need $E = 0$ so $\dot{x} = \sqrt{2V}$ and $x(t) \approx a - e^{-\omega t}$. This is the instanton (because it is like a soliton but in (Euclidean) time, so it is a lump at an instant) configuration. The configuration going from a to $-a$ is called an anti-instanton.

For the case $V = \lambda(x^2 - a^2)^2$, the instanton solution is $\bar{x}(t) = a \tanh(\frac{1}{2}\omega(t - t_0))$ where $\omega = 2a\sqrt{2\lambda/m}$ and t_0 is an example of a zero-mode of the solution, which is expected because of the time translation invariance.