215c, 4/22/20 Lecture outline. © Kenneth Intriligator 2020.

⋆ **Week 4 reading: Tong chapter 2.**

http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

• Last time: quantum effects can dramatically alter classical results. An example is the double well $V(x) = \lambda(x^2 - a^2)^2$ in QM, where classically the Z_2 symmetry is spontaneously broken, but quantumly it is restored thanks to tunneling. This will be discussed as a warmup for our case of interest, showing that although classically vacua have a choice of the winding number n in space, quantumly we must sum over them because of tunneling. We can roughly think about having classical vacua $|n\rangle$ and quantum tunneling transition configurations T_k such that $\langle n + k|T_k|n\rangle \neq 0$. Instead the vacua are labelled by $|\theta\rangle = \sum_{k=-\infty}^{\infty} e^{ik\theta}|n\rangle$ with θ a good quantum number (in that $\langle\theta'|e^{-iHt}\theta\rangle = 2\pi\delta(\theta' - \theta)\sum_k e^{ik\theta}\langle k|e^{-iHt}|0\rangle \sim \delta(\theta' - \theta)$). The path integral, $[dA_\mu]$ has to include all of the $\pi_3(G) = \mathbf{Z}$ topological winding sectors k and $e^{ik\theta}$ comes from the S_θ term, $e^{ik\theta}\langle k|e^{-iHt}|0\rangle = \int [dA_\mu]e^{iS_{YM+matter}+i\theta S_\theta}$.

Tunneling can be seen from classical solutions of the Euclidean theory's EOM, and these are called instantons or anti-instantons. For Euclidean Yang-Mills, these are the solutions of $F_{\mu\nu} = \pm{*F_{\mu\nu}}$ and they have $S_{YM} \to S_{inst} = \frac{8\pi^2}{g^2}|k|$ and $e^{-S} \to e^{-8\pi^2|k|/g^2+i\theta k}$. Mention another example: pair production of charged particles from the vacuum by an external electric field; again, this can be as patching into Minkowski spacetime a classical solution of the Euclidean EOM, which looks like a charged particle moving in a circle.

• Continue with the QM double well, $V = \lambda(x^2 - a^2)^2$ and noting that, although the classical groundstate spontaneously breaks the Z_2, quantum tunneling restores it: $|\pm\rangle = \frac{1}{\sqrt{2}}(|L\rangle \pm |R\rangle)$ have parity ± 1, with $E_\pm = E_0 \mp Ke^{-B}$, where K is a calculable constant. In the Euclidean path integral, the extremal tunneling solution comes from extrema of the classical Euclidean action. In the double well example, the classical minima at $x = \pm a$ become local maxima when $V \to -V$. There is then a classical solution that connects $|-a, -T/2\rangle$ to $\langle a, T/2|$. The Euclidean version of the path integral gives $K(x_f, T/2; x_i, -T/2) = \langle x_f|e^{-HT/\hbar}|x_i\rangle = N\int [dx]e^{-S/\hbar}$. Taking $H|n\rangle = E_n|n\rangle$, get $\sum_n e^{-E_nT/\hbar}\langle x_f|n\rangle\langle n|x_i\rangle$. We will be interested in large T, and then the sum is dominated by the states of lowest energy.

Evaluate the path integral by the usual method of time-slicing and approximating with a gaussian around the stationary path – which now is the instanton solution $\bar{x}(t)$ of the classical Euclidean EOM, i.e. the motion with $V \to -V$, so $E = \frac{1}{2}(\dot{\bar{x}})^2 - V(x)$ is a constant of the motion. Taking $T \to \infty$, we need $E = 0$ so $\dot{\bar{x}} = \sqrt{2V}$ and $\bar{x}(t) \approx a - e^{-\omega t}$. For the case $V = \lambda(x^2 - a^2)^2$, the instanton solution is $\bar{x}(t) = a\tanh(\frac{1}{2}\omega(t - t_0))$ where $\omega = 2a\sqrt{2\lambda/m}$ and t_0 is an example of a zero-mode of the solution, which is expected because of the time translation invariance. The configuration going from a to $-a$ is called an anti-instanton. The instanton's action is $S_{inst} = \int dt(\frac{1}{2}\dot{\bar{x}}^2 + V) = \int dt\dot{x}^2 = \int_{-a}^{a} dx\sqrt{2V}$. So the Euclidean path integral approximation will give e,g, $\langle a|e^{-HT}|-a\rangle = N\int_{x(0)=-a}^{x(T)=a}[dx(t)]e^{-S_E[x(t)]}$ and we take $x(t) = \bar{x}(t) + \delta x(t)$ to get $S_E = S_{inst} + \int d\tau\delta x\Delta\delta x + \mathcal{O}(\delta x^3)$ so get for the path integral $\approx \int_0^T dt_0 Je^{-S_{inst}}\frac{1}{\det'^{\,1/2}(\Delta)}$ with J a Jacobian determinant. The factor of $e^{-S_{inst}/\hbar}$ reproduces the familiar barrier penetration transmission amplitude coefficient $|T(E)| = \exp(-\frac{1}{\hbar}\int_{x_1}^{x_2} dx\sqrt{2(V-E)})(1 + \mathcal{O}(\hbar))$. Recall that the perturbative loop expansion is an expansion in powers of $\hbar$, whereas this effect is $\sim e^{-S_{inst}/\hbar}$, which does not have a Taylor expansion in $\hbar$ – it is non-perturbative.

There are a couple of technical subtleties that require care: the zero mode t_0 is a zero eigenvector of the operator in the determinant above, and has to be eliminated from the determinant and treated separately, as a "collective coordinate": it is treated as a quantum variable which, together with conjugate momentum is quantized. In the path integral description, we need to integrate over these collective coordinates. Also, there is not just a single instanton contribution but, instead, we need to sum over a dilute gas of instantons and anti-instantons, e.g. instead of just $-a \to a$, there is $-a \to a \to -a \to a$ etc. This leads to a series that can be summed. See Coleman's lectures and Tong's notes for more details. This could be a topic for a final presentation. The upshot is that the dilute instanton sum gives

$$\langle \pm a|e^{-HT/\hbar}|-a\rangle = \frac{1}{2}(\frac{\omega}{\pi\hbar})^{1/2}e^{-\omega T/2}[\exp(Ke^{-S_0/\hbar}T) \mp \exp(-Ke^{-S_0/\hbar}T)],$$

where the exp comes from the instanton dilute gas sum over even or odd numbers of instantons. Compare with $\langle \pm a|e^{-HT/\hbar}|-a\rangle = \sum_n e^{-E_nT/\hbar}\langle \pm a|n\rangle\langle n|-a\rangle$ to read off the energies of the two low-lying states, $E_\pm \approx \frac{1}{2}\hbar\omega \mp \hbar Ke^{-S_0/\hbar}$ where $E_\pm$ are the energy of the parity even and odd eigenstates.

• The basic instanton of an $su(2)$ gauge theory solves $F_{\mu\nu} = *F_{\mu\nu}$ with $A_\mu|_{r\to\infty} \to iU\partial_\mu U^{-1}$ where U has winding number $k = 1$, e.g. $U = x_\mu\sigma^\mu/\sqrt{x^2}$ with $\sigma^\mu = (1, -i\vec{\sigma})$. This leads to $A_\mu(x) = \frac{\eta^i_{\mu\nu}x^\nu\sigma^i}{x^2+\rho^2}$. The $\eta^i_{\mu\nu} = -\eta^i_{\nu\mu}$ are called 't Hooft matrices and given e.g. by $\eta^1_{12} = \eta^1_{34} = 1 = -\eta^1_{21} = -\eta^1_{43}$ (with other components zero), $\eta^2_{13} = \eta^2_{42} = 1$ (antisymmetrized, with others zero), and $\eta^3_{14} = \eta^3_{23} = 1$ (antisymmetrized, others zero).

The η^a form a representation (the adjoint) of $SU(2)$, and are self-dual (it's the adjoint rep), $\eta^a_{\mu\nu} = *\eta^a_{\mu\nu}$. If we think of the Euclidean rotation group as $SU(2)_L \times SU(2)_R$, the A^a_μ map $SU(2)_L \to SU(2)_{gauge}$ and preserve a $SU(2)_{L+gauge}$ diagonal subgroup – these give 3 Euler angle rotational zero modes of where a point in the S^3 maps to a point in G.

There are spacetime translation invariance zero modes: we can replace $x^\mu \to x^\mu - x^\mu_0$ where x^μ_0 is the spacetime location of the instanton. The parameter ρ gives the instanton size. It is a another zero mode because the classical equations are scale invariant. The $su(2)$ rotations give 3 more zero modes. All together the single instanton of $su(2)$ has 8 zero modes; for $su(N)$ and instanton number k this becomes $4Nk$. There is a lot more physics that could be discussed, and also connections to math.

The ρ collective coordinate needs to be integrated over, and this generally leads to IR divergences at large ρ. For some modified theories (e.g. if $su(2)$ is spontaneously broken to a $u(1)$ subgroup or completely broken) or modified quantities (higher-point correlation functions), the ρ integral becomes convergent and peaked at finite sized instantons, which helps to justify a dilute instanton gas approach. In other cases, the breakdown of the instanton approach reveals that it is not a good approximation to the path integral.

Let $|\theta\rangle$ denote a state with a given value of the θ angle, and it could be with $\theta = \langle ga\rangle$ where a is the axion and g is some dimensionful coupling. Then in the dilute gas approximation where we sum over n instantons and $\bar{n}$ anti-instantons, in a finite spacetime box of Euclidean volume VT, they contribute $\langle\theta|e^{-HT}|\theta\rangle \approx \sum_{n,\bar{n}}(Ke^{-S_0}VT)^{n+\bar{n}}e^{i(n-\bar{n})\theta}/n!\bar{n}! = \exp(2KVTe^{-S_0}\cos\theta)$, so $E(\theta)/V = -2K\cos\theta e^{-S_0}$, which lifts the perturbative degeneracy of the θ vacua, preserving of course $\theta \sim \theta + 2\pi$.

next time:

• Briefly mention pair creation in an external electric field. E.g. Schwinger calculated in 1951that the probability of pair creating particles of mass m and charge q in an external electric field E is $P = 1 - e^{-\gamma V}$ where γ (technically for spin 0) is $\gamma = \sum_{n=1}^\infty \frac{(-1)^{n+1}(qE)^2}{8\pi^3 n^2}e^{-\pi m^2 n/|qE|}$. Affleck, Alvarez, and Manton (1982) found a much simpler rederivation of this result using instantons and the worldline path integral for the charged particle in an external field: the electric field in Euclidean space resembles a magnetic field and the classical motion in a constant electric field $\vec{E} = E\hat{z}$ is circular, similar to a cyclotron orbit in the (z, x_4) Euclidean plane, with radius m/qE. The solution represents tunneling between the vacuum with zero particles and the vacuum with a particle-antiparticle pair. This could also be a nice topic for a final presentation.