Example: Free 2d boson:

$$S = \frac{1}{4\pi} \int d^2\sigma \left(\partial^a \phi \, \partial_a \phi - m^2 \phi^2 \right)$$

must take massless, $m=0$, for scale inv. thy.

In complex coords: $S = \frac{1}{4\pi} \int d^2 z \, \partial\phi \, \bar\partial \phi$

ϕ EOM: $\partial \bar\partial \phi = 0$ 2d wave eqn. True as operator statement except for contact terms from operators at coincident points. E.g.

$$\partial_z \partial_{\bar z} \langle \phi(z,\bar z) \, \phi(w,\bar w) \rangle = -2\pi \, \delta^2(z-w, \bar z - \bar w)$$

i.e. $\langle \phi \phi \rangle$ = Green's fn. for 2d wave eqn.

Reason:
$$0 = \int [d\phi] \frac{\delta}{\delta\phi(z,\bar z)} \left[e^{-S} \phi(w,\bar w) \right]$$

$$= \int [d\phi] \left(\frac{-\delta S}{\delta \phi(z,\bar z)} \phi(w,\bar w) + \frac{\delta\phi(w,\bar w)}{\delta\phi(z,\bar z)} \right) e^{-S}$$

$$= \frac{1}{2\pi} \partial_z \partial_{\bar z} \langle \phi(z,\bar z) \, \phi(w,\bar w) \rangle + \delta^2(z-w, \bar z - \bar w) \checkmark$$

Since $\partial\bar\partial \ln |z|^2 = 2\pi \, \delta^2(z,\bar z) \Rightarrow$

$$\boxed{\langle \phi(z,\bar z) \, \phi(w,\bar w) \rangle = -\ln |z-w|^2}$$

Since $\partial\bar\partial\phi = 0 \implies \phi = \phi_L(z) + \phi_R(\bar z)$

$$\boxed{\begin{aligned}\langle \phi_L(z)\,\phi_L(w)\rangle &= -\ln(z-w)\\ \langle \phi_R(\bar z)\,\phi_R(\bar w)\rangle &= -\ln(\bar z-\bar w)\end{aligned}}$$

$\langle \phi_L(z)\,\phi_R(\bar w)\rangle = 0$ etc.

$$\phi_L = \hat{x}_L - i\hat{p}_L \ln z + i \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m}\frac{\alpha_m}{z^m}$$

Fourier modes

$$[\phi, \pi] = i\delta(\sigma-\sigma') \qquad \pi = \frac{\delta \mathcal{L}}{\delta(\partial_\tau \phi)}$$

quantize: $\alpha_m \longrightarrow$ operators $(\alpha_0 \equiv \hat{p}_L)$

$[\hat{x}, \hat{p}] = i \qquad [\alpha_m, \alpha_n] = m\,\delta_{m+n,0}$

$\alpha_{n>0}|0\rangle = 0 \qquad \langle 0|\alpha_{n<0} = 0$

Can verify these give $\langle \phi_L(z)\,\phi_L(w)\rangle = -\ln(z-w)$

~~Ale~~ Can rescale $\alpha_{-m} = \sqrt{m}\, a^{\dagger}_{m} \quad m>0$

$\alpha_m = \sqrt{m}\, a_m \quad m>0$

$[a^{\dagger}_m, a_m] = 1$ usual kind of creation / annihilation ops

Find stress tensor by putting in h_{ab} & varying $\Rightarrow$

$$T_{zz} = -\frac{1}{2}(\partial\phi)^2 \qquad T_{\bar{z}\bar{z}} = -\frac{1}{2}(\bar{\partial}\phi)^2 \qquad T_{z\bar{z}} = 0$$

(classically)

OPE: $T(z)\,\phi(w) = -\frac{1}{2}(\partial\phi)^2\,\phi(w) \sim \left(-\frac{1}{2}\right)(2)\,\partial\phi\,\partial\phi\,\phi$

using $\phi(z)\,\phi(w) = -\ln(z-w)$

$$\partial\phi(z)\,\phi(w) = -\frac{1}{z-w}$$

$$\partial\phi(z)\,\partial\phi(w) = \frac{-1}{(z-w)^2} \qquad \text{etc}$$

so $T(z)\,\phi(w) = \dfrac{\partial\phi(w)}{(z-w)} + \text{nonsing.}$

expected formula for ϕ "primary" with $h=0$.

now check $T(z)\,\partial_w\phi(w) = \left(-\frac{1}{2}\right)(2)\,\partial\phi(z)\,\partial\phi(z)\,\partial\phi(w)$

$$= \frac{\partial\phi(z)}{(z-w)^2} + \text{nonsing} = \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial(\partial\phi(w))}{(z-w)} + \text{nonsing}$$

$\Longrightarrow$ $\partial\phi$ is a primary field of $h=1$

($\&\ \bar{h}=0$)

Define $i\partial_z\phi \equiv J_z \equiv J \qquad -i\bar{\partial}_{\bar{z}}\phi \equiv J_{\bar{z}} \equiv \bar{J}$

EOM $\Rightarrow$ $\partial_{\bar z} J = 0$ $\quad \partial_z \bar J = 0$

consered left & right moving currents.

Global symmetry. $\quad \langle J(z) J(w) \rangle = \dfrac{1}{(z-w)^2}$

J & $\bar J$ charge eigenstates $\quad e^{ip\phi_L(z) + i\bar p \phi_R(\bar z)}$

$$J(z)\, e^{ip\phi_L(w)} = (i\partial_z \phi_L)(ip\phi_L)\, e^{ip\phi_L}$$

$$= \frac{p}{(z-w)} e^{ip\phi_L(w)} \quad \text{so} \quad \underbrace{Q = \oint \frac{dz}{2\pi i} J_z}_{= \hat p} \text{ has}$$

eigenvalue $Q = \hat p = p$ on $e^{ip\phi_L}$

Check : $\quad T(z)\, e^{ip\phi_L(w)} = \dfrac{p^2/2}{(z-w)^2} e^{ip\phi_L(w)} +$

$+ \dfrac{\partial_w e^{ip\phi(w)}}{(z-w)} +$ nonsing $\Rightarrow e^{ip\phi_L}$ is primary

op of $(h, \bar h) = (\frac{p^2}{2}, 0)$. General

$e^{ip\phi_L + i\bar p \phi_R}$ has $(h, \bar h) = \left(\frac{p^2}{2}, \frac{\bar p^2}{2}\right)$.

Can also see from $\langle e^{ip\phi_L(z)} e^{-ip\phi_L(w)} \rangle$

$$= e^{p^2 \langle \phi_L(z) \phi_L(w) \rangle} = \frac{1}{(z-w)^{p^2}} \Rightarrow e^{\pm ip\phi_L}$$

$\uparrow$ Wicks thm or BCH $\qquad$ has $h = p^2/2$

OPE: $$e^{ip_1 \phi_L(z)} e^{ip_2 \phi_L(w)} = \frac{e^{i(p_1+p_2)\phi_L(w)}}{(z-w)^{-p_1 p_2}} + \cdots$$

Can gen'ly use Wicks thm to show

$$\langle \prod_{i=1}^{n} e^{ip_i \phi_L(z_i)} \rangle = e^{-\sum_{i<j} p_i p_j \langle \phi_L(z_i) \phi_L(z_j) \rangle}.$$

$$\cdot \langle e^{i\sum p_i \phi_L(z_i)} \rangle = \delta_{\sum p_i} \prod_{i<j=1}^{n} (z_i - z_j)^{p_i p_j}$$

" 0 if $\sum p_i \neq 0$

& 1 otherwise

combinatorics

Finally $$\langle T(z) T(w) \rangle = \left(-\tfrac{1}{2}\right)^2 2 \; :\partial\phi\partial\phi::\partial\phi\partial\phi:$$

$$= \frac{1/2}{(z-w)^4} \Rightarrow \text{ this thy has } c = 1$$

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} :\alpha_{m-n} \alpha_n: \quad \leftarrow \text{all lowering ops to left}$$

$(\alpha_0 \equiv p)$

$$\hookrightarrow L_0 = \frac{\hat{p}^2}{2} + \sum_{n=1}^{\infty} \alpha_{-n} \alpha_n$$

$[L_0, \alpha_n] = -n\,\alpha_n$ | $\alpha_{n>0}$ annihilation
$\alpha_{n<0}$ creation

Understand $H = L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}$ Casimir

energy: usual $\frac{1}{2}\hbar\omega$ with $\omega = n$

for each oscillator $\alpha_{\pm n}$

$\Rightarrow$ Groundstate energy is $\frac{1}{2}\sum_{n=1}^{\infty} n$

for each left & right mover.

Use $\sum_{n=1}^{\infty} n \equiv \zeta(-1)$ with $\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s}$

by continuation $\zeta(-1) = -1/12$

Primary op $e^{i(p_L\phi_L + \bar{p}_R\phi_R)}$ $\longrightarrow$

- primary state $|p_L, p_R\rangle$ eigenstate of

$J_z = i\partial\phi$ $\quad \bar{J}_{\bar{z}} = -i\bar{\partial}\phi$ conserved charges

$$Q_L = \oint \frac{dz}{2\pi i} J_z = \hat{P}_L \qquad Q_R = \hat{P}_R$$

$$\hat{P}_L\,|p_L, p_R\rangle = p_L\,|p_L, p_R\rangle$$

$\uparrow$ operator $\qquad \uparrow$ eigenvalue

- also L_0 eigenstate $\quad L_0|p_L, p_R\rangle = \frac{p_L^2}{2}|p_L, p_R\rangle$

General descendant state $\quad \prod_i \alpha_{-n_i}\bar{\alpha}_{-\bar{n}_i}\,|p_L, p_R\rangle$

again has $\hat{P}_L = p_L \quad \hat{P}_R = p_R$ eigenvalues

$$L_0 = \sum n_i + \frac{p_L^2}{2} \qquad \bar{L}_0 = \sum \bar{n}_i + \frac{p_R^2}{2}$$

- Note large #s of $\{n_i\}$ with $\sum_i n_i = N$ fixed. $p(N)$ = # partitions of N, huge for large N.

Another example: free left & right moving fermions

$$S = \frac{1}{4\pi}\int d^2z \, (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi})$$

ψ & $\bar{\psi}$ anti commuting

ψ : left moving, $h = 1/2, \bar{h} = 0$

$\bar{\psi}$ right moving, $\bar{h} = 1/2, h = 0$

$$\langle \psi(z) \psi(w) \rangle = \frac{1}{z-w} \qquad \langle \bar{\psi}(\bar{z}) \bar{\psi}(\bar{w}) \rangle = \frac{1}{\bar{z}-\bar{w}}$$

others vanish

$$T = -\frac{1}{2} \psi \partial \psi \qquad \bar{T} = -\frac{1}{2} \bar{\psi} \bar{\partial} \bar{\psi}$$

check $T(z)\psi(w) \longrightarrow \psi$ primary with $h = 1/2$

$T(z)T(w) \longrightarrow c = 1/2 \qquad \bar{T}\bar{T} \longrightarrow \bar{c} = 1/2$

Could also consider only ψ_L thy w/ $c_L = 1/2 \quad c_R = 0$, but can't consistently be coupled to gravity since $c_L \neq c_R$

A more general class of examples: bc sys.

$$S = \frac{1}{2\pi}\int d^2z \; b\bar{\partial}c$$ b, c anticommuting

where $h_b = \lambda$, $\bar{h}_b = 0$ $h_c = 1-\lambda$, $\bar{h}_c = 0$

again $\langle b(z)\, c(w)\rangle = \frac{1}{(z-w)} = \langle c(z)\, b(w)\rangle$

but $T = \partial b\, c - \lambda\, \partial(bc)$

to give $T(z)\, b(w) = \frac{\lambda\, b(w)}{(z-w)^2} + \frac{\partial b(w)}{(z-w)} + \cdots$

This T gives $\langle T(z)\, T(w)\rangle = \frac{c/2}{(z-w)^4}$

with $c = 1 - 3(2\lambda - 1)^2$ ← (* show this)

Also global current $j = -bc$ * find $\langle T(z) j(w)\rangle$

A special case which will be important later: $\lambda = 2$

$c = 1 - 3(9) = -26$ ← Central charge of Faddeev Popov ghosts.

Another example which will be useful later $\beta\gamma$

$$S = \frac{1}{2\pi}\int d^2z\, \beta\bar{\partial}\gamma$$ β, γ commuting with

β primary $(h,\bar{h}) = (\lambda, 0)$ γ primary $(h,\bar{h}) = (1-\lambda, 0)$

$$\beta(z)\gamma(w) \sim -\frac{1}{z-w} \qquad \gamma(z)\beta(w) \sim \frac{1}{z-w}$$

$$T = :(\partial\beta)\gamma: - \lambda\,\partial(\beta\gamma)$$ (* show this)

$$c = 3(2\lambda-1)^2 - 1$$ $\lambda = 3/2$ case will arise as Faddeev Popov ghosts for superstring.

Return to considering Polyakov action

$$S_P = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-\det h}\; h^{ab}\, \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$$

Take $g_{\mu\nu}$ = spacetime metric to be flat

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$ for $\mu = 0 \cdots D-1$

Take worldsheet metric $h_{ab} = e^\phi \eta_{ab}$

And Weyl scale ϕ away (no prob if $c_T = 0$)