Recall $S_{NG} \sim \int_\Sigma dA = \int_\Sigma d^2\sigma \sqrt{\det_{ab}(\partial_a X^\mu \partial_b X_\mu)}$

no metric needed.

Eliminate $\sqrt{\ }$ at cost of introducing metric h_{ab}

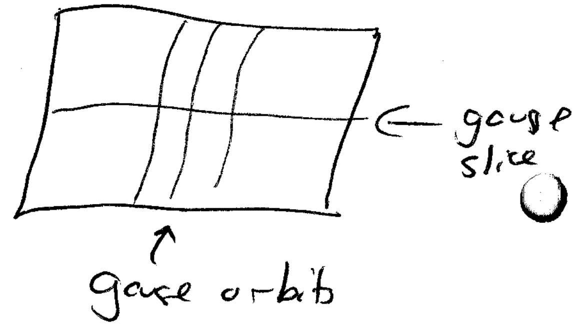
$$S_{WS} \sim \int d^2\sigma \sqrt{\det h}\, h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \cdots$$

$$\frac{\delta S}{\delta h_{ab}} = 0 \Rightarrow T_{ab} = 0 \quad \text{recover } S_{NG}$$

Imposed weaker condition $T_a{}^a = 0$ ← conf'l invt. Stronger $T_{ab} = 0$ cond → topological thy with general reparam inv. $\sigma^a \to \sigma'^a(\sigma)$.

~~Also~~ Ignored extra h_{ab} degrees of freedom since can eliminate 2 d.f. by $\sigma^a \to \sigma'^a$ & 3rd can be classically eliminated by $h_{ab} \to e^\phi h_{ab}$ Weyl rescaling. In quantum theory must do Feynman functional integral over h_{ab} modulo gauge transfs.

Faddeev Popov: field space



To avoid overcounting must divide out by gauge orbit symmetry. Restrict to gauge slice. Get Jacobian determinant.

Here gauge orbit from $(\sigma, \tau) \rightarrow (\sigma'(\sigma,\tau), \tau'(\sigma,\tau))$

reparam. symmetry. Inf. version acts on

h_{ab} by $h_{ab} \rightarrow h_{ab} + \nabla_a V_b + \nabla_b V_a$.

Fix by setting $h_{ab} = e^{\phi} \hat{h}_{ab}$ ← fixed ref metric eg η_{ab}

Variations about slice with $h_{zz} = h_{\bar{z}\bar{z}} = 0$

$$\delta h_{zz} = \nabla_z V_z \qquad \delta h_{\bar{z}\bar{z}} = \nabla_{\bar{z}} V_{\bar{z}}$$

so $$D h_{zz} \, D h_{\bar{z}\bar{z}} = \det \nabla_z \det \nabla_{\bar{z}} \, D V_z \, D V_{\bar{z}}$$

$$\int \frac{[D h_{ab}]}{\mathrm{Vol(orbit)}} = \int [d\phi] \, e^{-(c_m - 26) S_L} \, \frac{D V_z \, D \bar{V}_{\bar{z}}}{\mathrm{vol(orbit)}} \cdot$$

$$\cdot \det \nabla_z \det \bar{\nabla}_{\bar{z}}$$

$$\int \frac{DV_z \, D\bar{V}_{\bar{z}}}{\mathrm{Vol(orbit)}} = 1.$$

Get $\int [d\phi] \, e^{-(c_m - 26) S_L} \quad \det \nabla_z \det \bar{\nabla}_{\bar{z}}$

$\int [d\phi]$ decouples for $c_m = D = 26.$

Get $\det \nabla_z \det \nabla_{\bar{z}}$ from $\int [db\,dc] \, e^{-S_{ghost}}$

$$S_{ghost} = \frac{1}{2\pi} \int d^2z \left(b_{zz} \bar{\partial}_{\bar{z}} c^z + \bar{b}_{\bar{z}\bar{z}} \partial_z \bar{c}^{\bar{z}} \right)$$

b_{zz} : $(h, \bar{h}) = (2, 0)$ $\qquad$ c^z : $(h, \bar{h}) = (-1, 0)$

$\bar{b}_{\bar{z}\bar{z}}$ $(h, \bar{h}) = (0, 2)$ $\qquad$ $\bar{c}^{\bar{z}}$ $(h, \bar{h}) = (0, -1)$

$$b(z)\, c(w) \sim c(z)\, b(w) \sim \frac{1}{(z-w)}$$

This is the theory studied in an exercise,

with $c_{ghost} = -26.$

Write $\quad b(z) = \sum_{n=-\infty}^{\infty} \frac{b_n}{z^{n+2}} \qquad c(z) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n-1}}$

$$\overbrace{b(z)\, c}(w) \sim \frac{1}{z-w} \quad \Rightarrow \quad \{b_n, c_m\} = \delta_{n+m,0}$$

$$\{b_n, b_m\} = 0$$

$$\{c_n, c_m\} = 0$$

$$[db\,dc] = \prod_n db_n \prod_m dc_m$$

Recall anticommuting coordinates θ, $\theta^2 = 0$

$f(\theta) = f(0) + \theta f'(0)$ Simple Taylor exp.

$$\int d\theta\, 1 = 0 \qquad \int d\theta\, \theta = 1$$

so $\displaystyle \int db_n\, dc_n\, e^{-\lambda_n b_n c_n} = \lambda_n$

$$\int \prod_n db_n \prod_m dc_m\, e^{-M_{nm} b_n c_m} = \det M$$

why $\displaystyle \int [db][dc]\, e^{-S_{ghost}} = \det \nabla_z \det \nabla_{\bar{z}}$

The Mobius transformations = Conformal Killing vectors of the sphere worldsheet (incl. point $z=\infty$ to go from plane to sphere) correspond to normalizable zero modes of $c(z)$'s Dirac eqn

$$\bar{\partial} c(z) = 0 \qquad \partial \bar{c}(\bar{z}) = 0$$

give zero eigenvalues in $\det \overline{\nabla}_{\bar{z}}$ & $\det \nabla_z$ resp - must omit these i.e. soak up c & $\bar{c}$ zero modes to get non zero amplitudes.

Sol'n of $\bar{\partial} c(z) = 0$ is $c(z) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n-1}}$ $h=-1$

regularity at $z=0 \rightarrow$

$c_n |0\rangle_{gh} = 0$ for $n \geq 2$. Using $c_n^\dagger = c_{-n}$

${}_{gh}\langle 0| c_n = 0$ for $n \leq -2$

but c_1, c_0, c_{-1} need not annihilate $|0\rangle_{gh}$ or ${}_{gh}\langle 0|$ so these are the normalizable zero modos. Those correspond directly to the Mobius transformations L_{-1}, L_0, L_1

mapping $z \longrightarrow \lambda_{-1} + \lambda_0 z + \lambda_1 z^2$.

Does $b(z)$ have any zero modes?

$$b(z) = \sum_{n=-\infty}^{\infty} \frac{b_n}{z^{n+2}} \qquad b_n|0\rangle_{gh} = 0 \quad n \geq -1$$

$$_{gh}\langle 0| b_n = 0 \quad n \leq 1$$

no zero modes for $b(z)$ which are normalizable ~~at both~~ over entire sphere. All b_n annihilate either $_{gh}\langle 0|$ or $|0\rangle_{gh}$. Note $b_0|0\rangle_{gh} = 0$ and $_{gh}\langle 0| b_0 = 0$. Using $\{b_0, c_0\} = 1$

$$_{gh}\langle 0|0\rangle_{gh} = {}_{gh}\langle 0| \{b_0, c_0\} |0\rangle_{gh} = 0.$$

* Write $_{gh}\langle 0| c_0 |0\rangle_{gh} = {}_{gh}\langle 0| 1 \, c_0 |0\rangle_{gh}$

check that using $\{b_0, c_0\} = 1$ does <u>not</u> imply that the LHS vanishes but using $1 = \{b_1, c_{-1}\}$ would, as would using $1 = \{b_{-1}, c_1\}$

The non-vanishing amplitude is

$${}_{gh}\langle 0| c_{-1}\, c_0\, c_1 |0\rangle_{gh} = 1$$ i.e. we must

soak up the c_{-1}, c_0, c_1 zero modes to get a non-zero amplitude. For closed string must also soak up $\bar{c}_{-1}$, $\bar{c}_0$, $\bar{c}_1$

$${}_{gh}\langle 0| c_{-1}\bar{c}_{-1}\, c_0\bar{c}_0\, c_1\bar{c}_1 |0\rangle_{gh} = 1.$$

Gives $${}_{gh}\langle 0| c(z_1)\bar{c}(z_1)\, c(z_2)\bar{c}(z_2)\, c(z_3)\bar{c}(z_3) |0\rangle_{gh}$$

$$= |(z_1-z_2)(z_2-z_3)(z_3-z_1)|^2 \leftarrow \text{determinant factor}$$

seen before. So on sphere 3 of the physical states should $\longrightarrow$ $c(z)\bar{c}(\bar{z})V(z,\bar{z})$

& rest are $\longrightarrow$ $\int d^2z\, V(z,\bar{z})$. Both are conformal inv. iff $V(z,\bar{z})$ is primary op of $(h,\bar{h}) = (1,1)$ since $c\bar{c}$ & d^2z both transform as primary w/ $(h,\bar{h}) = (-1,-1)$

Let $j_z(z) = c^z b_{zz}$. Found in earlier exercise

$$T(z)\, j(w) = \frac{2\lambda - 1}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial j}{(z-w)}$$

here $\lambda = 2$ Call $2\lambda - 1 = Q$

Term $\frac{Q}{(z-w)^3} \Longrightarrow \bar{\partial} j = \frac{Q}{8\pi} \sqrt{\det h}\, R$

<u>anomaly</u>. ~~[illegible]~~ j gives c charge 1 & b charge -1 so anomaly means # c fields $-$ # b fields in vacuum must be non zero. Integrate $\int \bar{\partial} j = \frac{Q}{8\pi} \int \sqrt{\det h}\, R$

$\Longrightarrow$ (# c zero modes) $-$ (# b zero modes) $= \frac{1}{2} Q \chi$

$= Q(1-g)$ here $Q = 3$ so $3(1-L)$

$L =$ # of loops $=$ "genus g" of worldsheet.

For $g = 0$ agrees with what we found above.

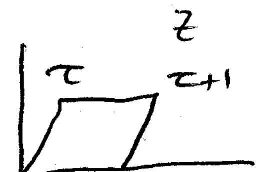
	# c zero modes (CKV)	# b zero modes
$g=0$	3	0
$g=1$	1	1
$g>1$	0	$3(g-1)$

$g=1$: 1 (c & $\bar{c}$) zero mode corresps to translation inv of insertion x  on torus.

$g>1$: no more conformal killing vectors.

b zero modes correspond to parameters needed to specify World sheet geometry which could not be eliminated by conformal transformations "moduli" of genus g Worldsheet.

E.g. $g=1$ torus $z \sim z+1 \sim z+\tau$



e.g. $\tau = \frac{iR_1}{R_2}$ ratio of lengths around the cycles. τ = 1 complex moduli of torus

Genus $g>1$: $3g-3$ complex moduli

Let $T^{tot} = T^{matter} + T^{ghost}$ stress tensor

Since T^{ghost} has $c = -26$, T^{tot} has $C_{tot} = C_{matter} - 26$

For $C_{matter} = D = 26$ vanishes.

$\rightarrow$ Liouville mode ϕ in $h_{ab} = e^{\phi}\eta_{ab}$ decouples in this case. Nice.

Before we had $|\psi\rangle_{phys}$ which should satisfy $L^{matter}_{n\geq 0}|\psi\rangle_{phys} = \delta_{n,0}|\psi\rangle_{phys}$

Now include the ghosts and write

$$|\psi\rangle_{\widetilde{phys}} = |\psi\rangle_{phys} \otimes c_1|0\rangle_{gh}$$

$(= c(z=0)|0\rangle_{gh})$

$L^{tot}_n = L^{matter}_n + L^{ghost}_n$ now we have

$$L^{tot}_{n\geq 0}|\psi\rangle_{\widetilde{phys}} = 0$$

$\left(\begin{array}{l} L^{M}_{0}|\psi\rangle_{phys} = |\psi\rangle_{phys} \\ L^{g}_{0}\, c_1|0\rangle_{gh} = -c_1|0\rangle_{gh} \end{array} \right.$

since $L^{M}_{0} + L^{ghost}_{0} \rightarrow (1-1) = 0$

Likewise $_{\widetilde{phys}}\langle \chi | L^{tot}_{n \le 0} = 0$

So effectively $T^{tot}(z) = \sum \frac{L_n^{tot}}{z^{n+2}}$

gives zero when put in physical state amplitudes. This is what we want for worldsheet reparameterization invariance.

BRST formalism: introduce operator Q_B with $Q_B^2 = 0$. Physical states must satisfy $Q_B |\psi\rangle = 0$. Since $Q_B^2 = 0$, any $|\psi\rangle = Q_B |\chi\rangle$ trivially satisfies $Q_B |\psi\rangle = 0$. These are null states & are orthogonal to physical states: decouple in phys processes

$$_{phys}\langle \psi_1 | \psi_2 \rangle_{null} = {}_{phys}\langle \psi_1 | Q_B | \chi_2 \rangle = 0$$

(take $Q_B = Q_B^\dagger$). Can always shift

$|\psi\rangle_{phys} \to |\psi\rangle_{phys} + Q|\chi\rangle$ for any

$|\chi\rangle$, get same inner prod with all physical states.

$\mathcal{H}_{closed}$ = space of $|\psi\rangle$ with $Q|\psi\rangle = 0$

$\mathcal{H}_{exact}$ = space of $Q|\chi\rangle$

$\mathcal{H}_{BRST} = \mathcal{H}_{closed} / \mathcal{H}_{exact}$

$\mathcal{H}_{exact}$ ~ gauge transformations so $\mathcal{H}_{BRST}$ are the physical states mod gauge ~~transfor~~ equivalences.

$\left(\int \frac{dz}{2\pi i}\left(cT_m + \frac{1}{2} cT_{gh} + \frac{3}{2}\partial^2 c\right)\right)$

Here we have a Q_B s.t. $\{Q_B, b_n\} = L_n^{tot}$

which satisfies $Q_B^2 = 0$ iff C_{tot} = $C_{matter} + C_{ghost} = 0$ i.e. $C_{matter} = D = 26$.

Physical states should satisfy

$Q_B |\psi\rangle_{\widetilde{phys}} = 0$ and $b_{n\geq 0} |\psi\rangle_{\widetilde{phys}} = 0$

↳ satisfied by $c_1|0\rangle_{gh}$ from before

then clearly

$L^{tot}_{n\geq 0} |\psi\rangle_{\widetilde{phys}} = 0$ using $\{Q_B, b_n\} = L_n^{tot}$.

Can prove no neg norm states in $\mathcal{H}_{BRST}$!

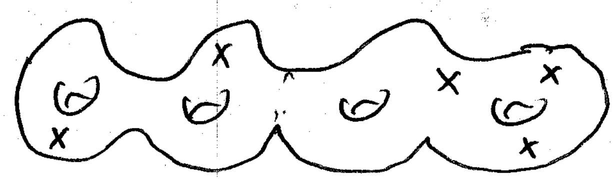
Physical states : $|\psi\rangle$ with $Q_B|\psi\rangle = 0$

- spurious states $Q_B|\chi\rangle$

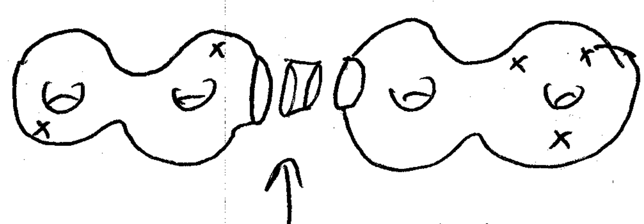
Physical operators $V(z,\bar{z})$ with $[Q_B, V] = 0$

Spurious operators $[Q_B, \chi(z,\bar{z})]$

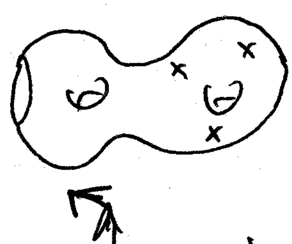
General amplitude



- factorize eg into



intermediate states

Q_B 

state living on boundary $\Rightarrow$

$$\oint \frac{dz}{2\pi i} j_B(z)$$

Q acts as contour integral on boundary. Deform contour off other end - can deform through operator insertions $\otimes$ since these are physical

- operators, $[Q_B, V] = 0$. Since contour can be pulled off end, Q annihilates the state on boundary

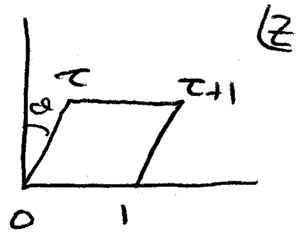
$\Rightarrow$ only physical states propagate in the intermediate channel (as we've already seen from OPE argument).

General features of loop amplitudes:

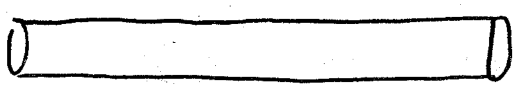
conformal transformation gage fixing doesn't fix all possible worldsheet metrics h_{ab} for genus g (= # handles in worldsheet) with $g > 0$.

For $g=1$ must still integrate over a 1 complex dimensional space of possible $\hat{h}_{ab}$. For $g > 1$ must integrate over a $3g-3$ dimensional "moduli space" of genus g Riemann surfaces. This is the meaning of the $b_{zz}(z)$ & $\bar{b}_{\bar{z}\bar{z}}(\bar{z})$ ghost zero modes.

For $g = 1$ torus, $z \sim z+1 \sim z+\tau$



$\tau =$ complex modulus of torus can't be eliminated by conf'l transf.

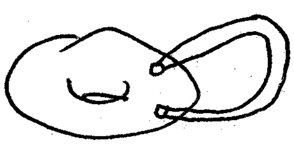


$\leftarrow$ scale this period from L_1 to 1 by scale transf.

$|\tau| = L_2/L_1$

glue ends together with twist by θ : $\tau = i|\tau|e^{-i\theta}$

<u>$g=2$</u> start with torus & add a handle



Extra complex parameters:
1 from length of tube & twist angle. 2 more from locations of two ends on original torus. But the location of one end is arbitrary from conformal killing vector of translation inv on torus. So use 1 end to fix the CKV & get 2 more complex moduli from loc. of other end & length/twist

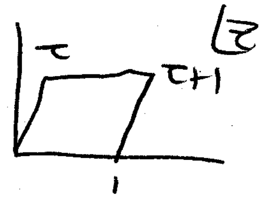
total $= 1+2 = 3$. 3 b zero modes

& no more c zero modes.

$\underline{g>2}$ Each extra handle gives 3 more complex moduli: 2 end locations + size & twist of tube

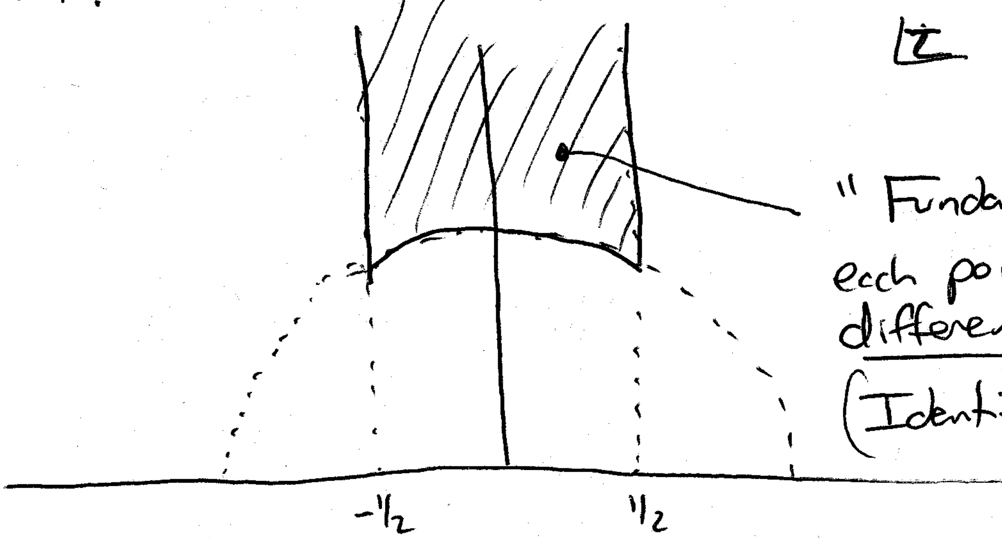
so $3(g-1)$ complex parameters. Agrees with # of b & $\bar{b}$ zero modes.

These moduli spaces of metrics on $g \geq 1$ surfaces are complicated spaces, with some nontrivial identifications. E.g. $g=1$



Can shift $\tau \to \tau+1$ & get same lattice. Can also take $\tau \to -1/\tau$.

E.g. for $\tau = iL_2/L_1$, this is just $L_1 \leftrightarrow L_2$ relabelling. These generate $\tau \to \frac{a\tau+b}{c\tau+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, i.e. $ad-bc=1$ a,b,c,d <u>integers</u>. $\tau \to \tau+1 \Rightarrow$ can take $|\mathrm{Re}(\tau)| \leq 1/2$. $\tau \to -1/\tau \Rightarrow$ can take $|\tau|>1$. Can also take $\mathrm{Im}\,\tau > 0$



"Fundamental domain" each point is a <u>different</u> torus. (Identify edges.)