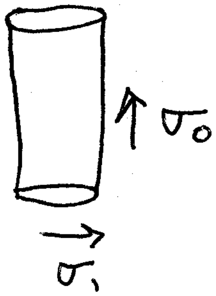


$\sigma_1 \sim \sigma_1 + 2\pi$ (Closed string)

$$\mathrm{Tr}\, e^{-\beta H} = \sum_a \langle a | e^{-\beta H} | a \rangle$$

= Field theory with periodic Euclidean time

$\sigma_0 \sim \sigma_0 + \beta$ define $\beta \equiv 2\pi \tau_2$

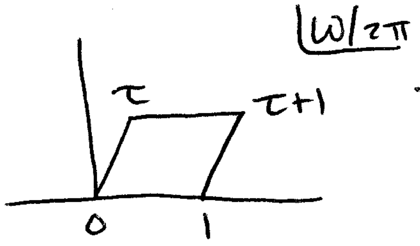
Can also consider $\mathrm{Tr}\, e^{2\pi i (\tau_1 P + i \tau_2 H)}$

$2\pi\tau_1$ = twist angle

$P = \frac{\partial}{\partial \sigma_1}$ generator $\quad H = \frac{\partial}{\partial \sigma_0}$ generator

$\sigma_1 + i\sigma_0 \equiv w \sim w + 2\pi n + 2\pi m \tau$

n, m arbitrary integers $\quad (\tau \equiv \tau_1 + i\tau_2)$



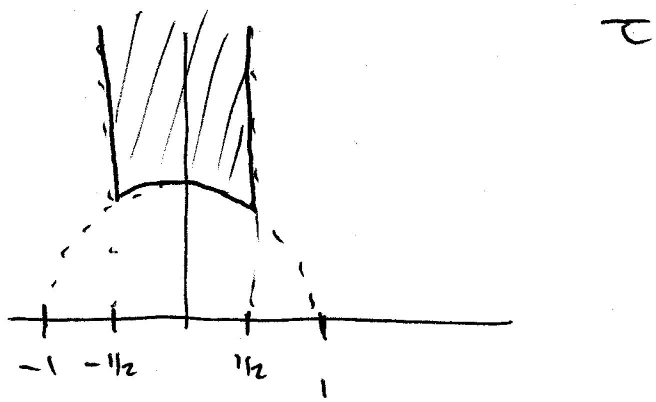
→ torus. Note $T: \tau \to \tau + 1$ gives same torus (rotate twist angle by 2π). $S: \tau \to -1/\tau$ also gives same torus (rename $\sigma_1 \leftrightarrow \sigma_0$)

* show $(ST)^3 : \tau \to \tau$, $w \to -w$

Together generate $\tau \to \dfrac{a\tau+b}{c\tau+d}$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

i.e. a, b, c, d integers with $ad - bc = 1$

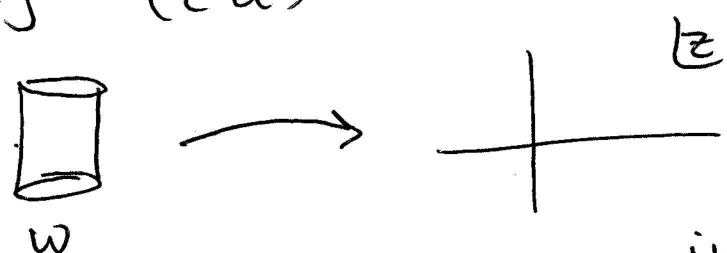
Can $\therefore$ take τ in Fundamental domain



Amplitudes $Z(\tau)$ must be modular invariant: $Z(\tau) = Z\left(\dfrac{a\tau+b}{c\tau+d}\right)$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$.

Map



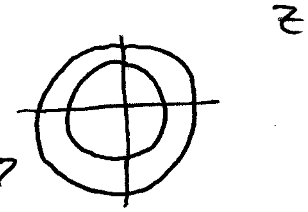
$z = e^{-iw}$

$w \sim w + 2\pi n \quad \longrightarrow \quad$ well defined under $z \to e^{2\pi i} z$

$w \sim w + 2\pi m \tau \quad \longrightarrow \quad$ identify $z \sim e^{-2\pi i \tau} z$

Can take $\quad 1 \le |z| \le e^{2\pi \tau_2}$

identify inside & outside edges to
get torus



$$\mathrm{Tr}\, e^{2\pi i(\tau_1 P + i\tau_2 H)} \to \mathrm{Tr}\, q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - \bar{c}/24}$$

$q \equiv e^{2\pi i \tau}$ $\qquad P = L_0 - \bar{L}_0$

$\bar{q} \equiv e^{-2\pi i \bar{\tau}}$ $(= q^*)$ $\qquad H = L_0 + \bar{L}_0 - \left(\frac{c + \bar{c}}{24}\right)$

(recall from $\frac{c}{12}\{w, z\}$)

For free string in D spacetime dimensions

$$Z(\tau) \equiv \mathrm{Tr}\, q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - \bar{c}/24}$$

$$= (\mathrm{Vol}\, D)\,(q\bar{q})^{-D/24} \int \frac{d^D k}{(2\pi)^D}\, (q\bar{q})^{\alpha' k^2/4} .$$

$$\cdot \prod_{\mu=1}^{D} \prod_{n=1}^{\infty} \sum_{l_{\mu n}=0}^{\infty} \sum_{\bar{l}_{\mu n}=0}^{\infty} q^{n l_{\mu n}}\, \bar{q}^{n \bar{l}_{\mu n}}$$

Where we are taking the Tr over all

states $\prod_{\mu=1}^{D} \prod_{n=1}^{\infty} (\alpha_{-n}^{\mu})^{l_{\mu,n}} (\cdot \text{ cc}) |k\rangle$

which has $L_0 = \frac{\alpha'}{4} k^2 + \sum_{\mu=1}^{D} \sum_{n=1}^{\infty} n\, l_{\mu,n}$

& likewise for $\bar{L}_0$.

$l_{\mu,n}$ = occupation # of the (μ,n) oscillator.

Sum over all $l_{\mu,n}$: $\sum_{l_{\mu,n}=0}^{\infty} q^{n l_{\mu,n}} = \frac{1}{1-q^n}$

So get $Z(\tau) = V_0 K(0) (q\bar{q})^{-D/24} \int \frac{d^D k}{(2\pi)^D} (q\bar{q})^{\alpha' k^2/4}$

$$\cdot \prod_{\mu=1}^{D} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \frac{1}{(1-\bar{q})^n}$$

Use Dedekind eta fn:

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$$

$$Z_{matter}(\tau) = i \overbrace{\mathrm{Vol}(D)}^{\text{normalization factor}} \cdot \underbrace{\int \frac{d^D k}{(2\pi)^D}}_{\text{usual factor from } \vec{p} = \frac{2\pi \vec{n}}{L} \text{ in box}} e^{-\pi \tau_2 \alpha' k^2} |\eta|^{-2D}$$

do k integral to get

$$Z_{matter} = i \,\mathrm{Vol}(D) \; Z_X(\tau)^D$$

$$Z_X(\tau) = (4\pi^2 \alpha' \tau_2)^{-1/2} \; |\eta(\tau)|^{-2}.$$

$Z_X(\tau)$ is invariant under $\tau \to \tau + 1$.

Also invt under $\tau \to -1/\tau$ using $\eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau)$ (Math identity).

Ghost partition function: must soak up $b_0, \bar{b}_0$, $c_0, \bar{c}_0$ zero modes. Also require ghost fermions to be periodic under shifting Euclidean time by $2\pi\tau_2$. $Z_{ghost}(\tau) =$

$$\mathrm{Tr}\left[(-1)^F c_0 b_0 \bar{c}_0 \bar{b}_0 \, q^{L_0 - c/24} \, \bar{q}^{\bar{L}_0 - \bar{c}/24}\right]$$

$$= (q\bar{q})^{1/12} \prod_{n=1}^{\infty} |1-q^n|^4 = |\eta(\tau)|^4$$

$\uparrow$ $b_n, c_n, \bar{b}_{-n}, \bar{c}_{-n}$ on $c_1 \bar{c}_1 |0\rangle_{gh}$

Unlike Z_X, Z_{ghost} is not quite modular invt under $\tau \to -1/\tau$. $\tau_2 Z_{ghost}(\tau)$ would be modular invt.

$Z_{ghost}(\tau)$ is not quite modular invt. because we are not quite done with the ghosts, 2 things remain:

1) $c_0, \bar{c}_0$ zero modes $\longrightarrow$ integrate $c(z)\bar{c}(\bar{z})$ over torus & divide by volume of torus to average over position of $c_0, \bar{c}_0$. Get factor of $\frac{1}{\tau_2}$

Since volume$(T^2) = \begin{vmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{vmatrix} = \tau_2$.

2) $b, \bar{b}$ zero modes $\longrightarrow$ integrate $\int_F d^2\tau$

So the ghosts give

$$\underbrace{\int d^2\tau}_{b \text{ z.m.}} \underbrace{\left(\frac{1}{\tau_2}\right)}_{c, \text{ z.m.}} \underbrace{Z_{ghost}(\tau)}_{\text{non z.m.}} = \underbrace{\int \frac{d^2\tau}{\tau_2^2}}_{\text{mod invt.}} \underbrace{\left(\tau_2 Z_{ghost}(\tau)\right)}_{\text{mod. invt.}}$$

The final result of ghosts + matter is $\therefore$

$$\int \frac{d^2\tau}{\tau_2^2} \left(\tau_2 Z_{ghost}\right) Z_M \sim \int \frac{d^2\tau}{\tau_2^2} Z_X(\tau)^{D-2}.$$

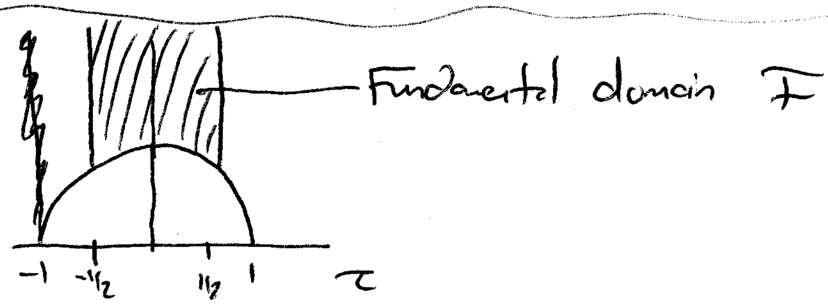
The effect of $\tau_2 Z_{ghost}$ is to shift $D \to D-2$. This is exactly what we've already seen from physical state condition:

V primary with $h=1$ eliminates 1 polarization & mod null states (gauge inv.) eliminates another $\to$ $D \to D-2$. Also fits with light cone description, with -2 related to the 2 reparametrizations $(\sigma, \tau) \to (\sigma'(\sigma,\tau), \tau'(\sigma,\tau))$ of 2d worldsheet.

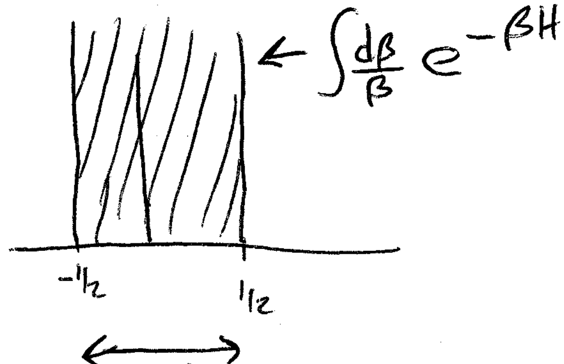
So for $D = 26$ we get $D-2 = 24$ independent oscillators, with 1 loop amp

$$\sim \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \left(\frac{1}{\tau_2^{1/2}\, \eta(\tau)\, \eta(\bar{\tau})} \right)^{24} .$$

For strings



For point particles



$\int d\sigma$ to project onto spinless states

$\beta \to 0 \Leftrightarrow T \to \infty$ UV region is not in fundamental domain! No UV divergences in string loop amplitudes since UV eliminated!

Happens because $\tau \to -1/\tau$ relates UV to IR. Can only get $\tau \to i\infty$ IR divergences!

Compactification of Spacetime

- Consider a worldsheet scalar ~ spacetime coordinate X with periodic identification

$$X \simeq X + 2\pi R \qquad S_{ws} = \int \frac{d^2z}{2\pi\alpha'} \partial X \bar{\partial} X$$

Vertex ops e^{ipX}, now $p = n/R$ quantized to be inv't under $X \simeq X + 2\pi R$. As in pt. particle phys with $n =$ integer.

- But can also have $X(\sigma + 2\pi, \tau) = X(\sigma, \tau) + 2\pi R w$ for arbitrary "winding number" integer w. Strings can wind around circle in spacetime.

$$X = X_L(z) + X_R(\bar{z})$$

$$\langle X_L(z) X_L(w) \rangle = -\frac{\alpha'}{2} \ln(z-w)$$
$$\langle X_R(\bar{z}) X_R(\bar{w}) \rangle = -\frac{\alpha'}{2} \ln(\bar{z}-\bar{w})$$

$$X_L = \hat{X}_L - i\hat{p}_L \frac{\alpha'}{2} \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\alpha_m}{m z^m}$$

$$X_R = \hat{X}_R - i\hat{p}_R \frac{\alpha'}{2} \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\bar{\alpha}_m}{m \bar{z}^m}$$

- Under $\sigma \to \sigma + 2\pi \qquad z \to e^{2\pi i} z$

$$X = X_L + X_R \rightarrow X + \frac{2\pi\alpha'}{2}\left(\hat{P}_L - \hat{P}_R\right) \overset{!}{=} X + 2\pi R w$$

So $\hat{P}_L - \hat{P}_R = \dfrac{2Rw}{\alpha'}$ & $\hat{P} = \dfrac{\hat{P}_L + \hat{P}_R}{2} = \dfrac{n}{R}$

so

$$\left.\begin{aligned} P_L &= \frac{n}{R} + \frac{wR}{\alpha'} \\ P_R &= \frac{n}{R} - \frac{wR}{\alpha'} \end{aligned}\right\}$$ eigenvalues of separate left & right conserved charges $\hat{P}_L$ & $\hat{P}_R$

The conserved charges $\hat{P}_L$ & $\hat{P}_R$ are associated with the conserved currents

$$J_L(z) = \frac{i2}{\alpha'}\partial X \qquad J_R(\bar{z}) = \frac{i2}{\alpha'}\bar{\partial} X$$

$\left(\frac{2i}{\alpha'}\right) \rightarrow$

which are separately conserved: $\bar{\partial} J_L = 0$, $\partial J_R = 0$

(thanks to $\partial\bar{\partial} X = 0$ EOM)

J_L has $(h, \bar{h}) = (1, 0)$ & J_R has $(h, \bar{h}) = (0, 1)$