Now move up to 1+1 dimensions.

In 0+1 d we had $Q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2)$

Q_1 & Q_2 = real (Hermitian). $Q^2 = (Q^\dagger)^2 = 0$

& $\{Q, Q^\dagger\} = 2H$ can be re-written as

$$\{Q_I, Q_J\} = 2H\,\delta_{IJ} \qquad I, J = 1, 2.$$

The 2d version of this is:

$$\left.\begin{aligned} \{Q_L, Q_L\} &= 2H_L \equiv 2(H-P) \\ \{Q_R, Q_R\} &= 2H_R \equiv 2(H+P) \\ \{Q_L, Q_R\} &= 0 \end{aligned}\right\}$$

$H \to$ energy
$P \to$ momentum
reduces to above for $P = 0$.

$$Q_L^\dagger = Q_L \qquad Q_R^\dagger = Q_R$$

Superspace:

$$Q_L = \frac{\partial}{\partial\theta_L} + \theta_L \overbrace{\left(i\frac{\partial}{\partial t} - i\frac{\partial}{\partial x}\right)}^{= H_L}$$

$$Q_R = \frac{\partial}{\partial\theta_R} + \theta_R \underbrace{\left(i\frac{\partial}{\partial t} + i\frac{\partial}{\partial x}\right)}_{= H_R}$$

$$D_L = \frac{\partial}{\partial\theta_L} - i\theta_L\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right), \quad D_R = \frac{\partial}{\partial\theta_R} - i\theta_R\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$$

$$S = \int d\theta_L d\theta_R \left(-\frac{1}{2} D_L \Phi D_R \Phi + W(\Phi) \right).$$

w/ $\Phi = \phi + i\theta_L \psi_L + i\theta_R \psi_R + \theta_L \theta_R F$

all fields here are real.

* Do the θ integrals to find the Lagrangian. Write the kinetic terms in terms of

$$\partial_L \equiv \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \quad \text{and} \quad \partial_R = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}.$$

In Euclidean space we write $Q_L \to Q$, $Q_R \to \bar{Q}$, again with Q & $\bar{Q}$ Hermitian, $\psi_L \to \psi$, $\psi_R \to \bar{\psi}$, $\partial_L \to \frac{\partial}{\partial z} \equiv \partial$, $\partial_R \to \frac{\partial}{\partial \bar{z}} \equiv \bar{\partial}$ $\quad z = i(x - i\tau) = \tau + ix$.

Consider $W = \Phi^K + a\Phi^{K-1} + \cdots$

What is $\mathrm{Tr}(-1)^F$? Make space a big circle no difference. Now ~~...~~ make circle small, $\mathrm{Tr}(-1)^F$ is unchanged. Get back susy QM.

$\mathrm{Tr}\,(-1)^F$ can always be found by reducing back to susy Q.M. (Though this method is not always useful in practice.) Here we know the susy QM answer: $\mathrm{Tr}(-1)^F = \begin{cases} 0 & \text{for } K \text{ odd} \\ \pm 1 & \text{for } K \text{ even} \end{cases}$

(In other cases we might not know susy QM answer)

E.g. $W(\Phi) = \frac{1}{3}\Phi^3 + a\Phi$ has $\mathrm{Tr}(-1)^F = 0$.

Classically: $a > 0 \Rightarrow$ broken susy (spontaneously)
groundstate at $\phi = 0$

$a < 0 \Rightarrow$ unbroken susy groundstates at
$\phi = \pm\sqrt{-a}$

In 0+1 susy broken in $a<0$ case too via instanton.

In 1+1 We'll see the classical answer is accurate.

Take e.g. $a = -1$.

Instanton: $\phi_{cl}(\tau) = \tanh(\tau - \tau_0) \leftarrow (x \text{ indep.})$

Soliton: $\phi_{cl}(x) = \tanh(x - x_0) \leftarrow (\tau \text{ indep.})$

The fact that the instanton is x indep means that its action $\sim L \equiv$ volume of space.

$$S_{cl} = \int dx\, d\tau\, \mathcal{L}[\phi_{cl}(\tau)] = L\,|\Delta W|$$

So instanton processes $\sim e^{-\frac{1}{\hbar} S_{cl}} \sim e^{-\frac{L}{\hbar}|\Delta W|}$

$\rightarrow 0$ for $L \rightarrow \infty$. Instanton no longer contributes in infinite volume.

In QM the wave function is symmetric around $\langle \phi \rangle = 0$. $\phi \rightarrow -\phi$ symmetry is unbroken

In 1+1 d QFT, Get two groundstates either centered at $\langle \phi \rangle = 1$ or at $\langle \phi \rangle = -1$

$\phi \rightarrow -\phi$ discrete symmetry is spontaneously broken. No tunnelling. No massless goldstino
$\hookrightarrow$ susy not broken

On the other hand, there is a soliton kink:

$\phi_{cl} = \tanh(x - x_0)$ $\leftarrow$ at rest at $x = x_0$.
Could boost to arbitrary velocity.

(Even in Minkowski space) ↑

The soliton ~ particle of mass

$$m = \int dx\, \mathcal{L}[\phi_{cl}] = |\Delta W|$$ ← Same calculation as in 0+1d for action

Check dimensional analysis:

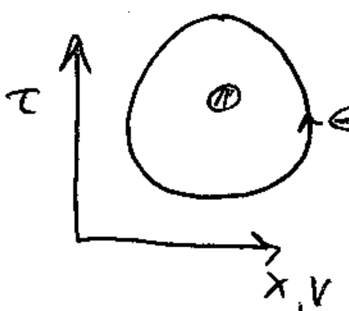
$[d^2x] = -2 \quad [d^2\theta] = +1 \quad$ so $[W] = 1$ ✓

More generally in d dimensional analog we would have

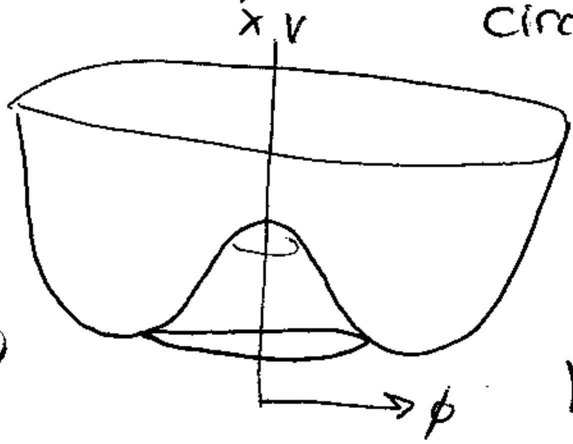
Kink = codimension 1 = $d-1-1$ brane

- $d=1 \rightarrow -1$ brane = instanton
- $d=2 \rightarrow$ 0 brane = soliton particle
- $d=3 \rightarrow$ 1 brane = string
- $d=4 \rightarrow$ 2 brane = domain wall

To get a finite action instanton in $d=2$ need a codimension 2 configuration.



← big circle (at ∞) surrounding instanton. Need fields to vary around circle, e.g.



complex ϕ can wind around minimum. Classically $\langle\phi\rangle \neq 0$ breaks $\phi \rightarrow e^{i\alpha}\phi$ symmetry.

Tunnelling restores symmetry in 1+1 dimensions.

Continuous symmetries not spontaneously broken in 1+1 dimensions (Coleman Mermin Wagner Thm)

Massless scalars in 1+1 d not Goldstone bosons.

E.g. whereas $S = \int d^2x \, d\theta_L \, d\theta_R \left(-\frac{1}{2} D_L \Phi D_R \Phi + W \right)$

for $W(\Phi) = \frac{1}{3}\Phi^3 - \Phi$ has 2 vacua

(breaking discrete $\phi \to -\phi$ symmetry)

$$S = \int d^2x \, d\theta_L \, d\theta_R \left(-\frac{1}{2} g_{IJ}(\Phi) \, D_L \Phi^I \, D_R \Phi^J \right)$$

has 1 vacuum, ~~spread over~~ with wavefunction spread over entire σ-model manifold M.

Above 1+1 dimensions even this tunnelling among degenerate vacua does not occur. Tunnelling costs energy but gains in entropy. In higher dimensions energy costs no longer compensated by entropy gains.