2/26/07 Lecture 13 outline

- Last time: Renormalized and bare Greens functions.

$$\tilde{\Gamma}_B^{(n)}(p_1, \dots p_n; \lambda_B, m_B, \epsilon) = Z_\phi^{-n/2} \tilde{\Gamma}_R^{(n)}(p_1, \dots p_n; \lambda_R, m_R, \mu, \epsilon).$$

For fixed physics, the LHS is some fixed quantity. The RHS depends on the renormalization point μ and the scheme. The LHS does not! This leads to what is known as the renormalization group equations, which state how the renormalized quantities must vary with μ. Rewrite above as

$$Z_\phi^{n/2} \tilde{\Gamma}_B^{(n)}(p_1, \dots p_n; \lambda_B, m_B, \epsilon) = \tilde{\Gamma}_R^{(n)}(p_1, \dots p_n; \lambda_R, m_R, \mu, \epsilon).$$

Now the RHS is finite, so the LHS must be too. So we can take $\epsilon \to 0$ without a problem.

- Before getting intro the renormalization group, let's take a little detour. Recall that

$$\int d^4x e^{ipx} \langle \Omega | T\phi(x)\phi(0) | \Omega \rangle = \frac{i}{p^2 - m^2 - \Pi'(p^2) + i\epsilon}.$$

By inserting a complete set of states,

$$\mathbf{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^2} \frac{1}{2E_p} |p\rangle\langle p|$$

the LHS can be written as

$$\int d^4x e^{ipx} \langle \Omega | T\phi(x)\phi(0) | \Omega \rangle = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon},$$

where

$$\rho(M^2) = \sum_\lambda 2\pi\delta(M^2 - m_\lambda^2) |\langle \Omega | \phi(0) | \lambda \rangle|^2 > 0$$

is the Kallen-Lehmann spectral density. Find $\rho(M^2) = 2\pi\delta(M^2 - m^2)Z$ for $M^2 \ll 4m^2$. For M^2 slightly below $4m^2$ there are new delta functions, at the bound states. Starting at $4m^2$, $\rho(M^2)$ is some positive function. This implies that

$$\frac{i}{p^2 - m^2 - \Pi'(p^2) - i\epsilon} = \frac{iZ}{p^2 - m^2 - i\epsilon} + \int_{\sim 4m^2}^\infty \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon}.$$

The LHS has a simple pole, with residue iZ, at $p^2 - m^2$. Then there can be a few more simple poles, for p^2 slightly below $4m^2$.

Starting at $p^2 = 4m^2$, there is a branch cut, corresponding to producing two more more free particles. Note $\mathcal{M}(s) = \mathcal{M}(s^*)^*$ implies that the real part of $\mathcal{M}$ is continuous across the cut, but the imaginary part can be discontinuous: $Im\mathcal{M}(s+i\epsilon) = -Im\mathcal{M}(s-i\epsilon)$. We'll return to this shortly.

The above equality, back in position space and taking $\partial/\partial t$, leads to the equal time commutators, $[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y})$, matching the coefficient of the delta function on the two sides of the resulting equation gives

$$1 = Z + \int_{\sim 4m^2}^{\infty} \frac{dM^2}{2\pi}\rho(M^2) \geq Z.$$

So $Z \leq 1$, with $Z = 1$ iff the theory is a free field theory. Let's compare with what we found last lecture,

$$\delta_Z^{(2)} = -\frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{\epsilon}.$$

Looks good and negative (for $\epsilon > 0$).

• LSZ (Lehmann, Symanzik, Zimmermann '55). Long discussion (see e.g. Peskin). Let's just state the result: the S-matrix element for m incoming particles and n outgoing ones is given by

$$\langle \mathbf{p_1} \ldots \mathbf{p_n} | S | \mathbf{k_1} \ldots \mathbf{k_m} \rangle = \lim_{o.s} \prod_{i=1}^{n} (p_i^2 - m_i^2) Z_i^{-1/2} \prod_{j=1}^{m} (k_j^2 - m_j^2) Z_j^{-1/2} \tilde{G}^{n+m}(-p_i, k_i).$$

Here $\tilde{G}^{n+m}$ is the full $n+m$ point Green's function, including disconnected diagrams etc. The limit is where we take the external particles on shell. In this limit, the $p_i^2 - m_i^2$ and $k_j^2 - m_j^2$ prefactors all go to zero. These zeros kill everything on the RHS except for the connected contributions to $\tilde{G}$. Accounting for the fact that we amputate the external propagators, which go like $iZ_i(p_i^2 - m_i^2)^{-1}$, the above becomes

$$\langle \mathbf{p_1} \ldots \mathbf{p_n} | S | \mathbf{k_1} \ldots \mathbf{k_m} \rangle = Z^{(n+m)/2} \tilde{G}^{n+m}_{conn,B}(-p_i, k_i) = \tilde{G}^{n+m}_{conn,R}(-p_i, k_j)$$

This is the promised general relation between the connected Greens functions (and in particular $\tilde{\Gamma}$) and S-matrix elements.