2/28/07 Lecture 14 outline

• Last time: LSZ

$$\langle \mathbf{p_1}\dots\mathbf{p_n}|S|\mathbf{k_1}\dots\mathbf{k_m}\rangle = \lim_{o.s}\prod_{i=1}^{n}(p_i^2-m_i^2)Z_i^{-1/2}\prod_{j=1}^{m}(k_j^2-m_j^2)Z_j^{-1/2}\tilde G^{n+m}(-p_i,k_i).$$

Here $\tilde G^{n+m}$ is the full $n+m$ point Green's function, including disconnected diagrams etc. The limit is where we take the external particles on shell. In this limit, the $p_i^2-m_i^2$ and $k_j^2-m_j^2$ prefactors all go to zero. These zeros kill everything on the RHS except for the connected contributions to $\tilde G$. Accounting for the fact that we amputate the external propagators, which go like $iZ_i(p_i^2-m_i^2)^{-1}$, the above becomes

$$\langle \mathbf{p_1}\dots\mathbf{p_n}|S|\mathbf{k_1}\dots\mathbf{k_m}\rangle = Z^{(n+m)/2}\tilde G^{n+m}_{amp,conn,B}(-p_i,k_i)=\tilde G^{n+m}_{amp,conn,R}(-p_i,k_j)$$

This is the promised general relation between the amputated, connected Greens functions (and in particular $\tilde\Gamma$) and S-matrix elements.

• Example from last quarter: tree-level contribution to the Compton effect, scattering light off an electron. The S matrix element is given at tree-level by $S=1+iT$, where

$$\langle f|iT|i\rangle = i(2\pi)^4\delta^4(k_f+p_f-k_i+p_i)\mathcal{M}_{fi}$$

$$\mathcal{M}_{fi}=-e^2\bar u(p_f,\alpha_f)\left(\not\epsilon_f\frac{1}{\not p_i+\not k_i-m}\not\epsilon_i+\not\epsilon_i\frac{1}{\not p_i-\not k_f-m}\not\epsilon_f\right)u(p_i,\alpha_i).$$

More generally, the S-matrix element is given according to LSZ by the connected, amputated Greens functions. Note that it is not just the 1PI diagrams contributing (the above example is a non-1PI contribution).

• Optical theorem. The S-matrix $S=U(t_f=\infty,t_i=-\infty)$ is unitary, $S^\dagger S=1$. Write $S=1+iT$, then get $2Im(T)\equiv -i(T-T^\dagger)=T^\dagger T$. Thus

$$-i(2\pi)^4\delta^4(p_f-p_i)(\mathcal{M}_{fi}-\mathcal{M}^*_{if})=\sum_m\prod_i\int\frac{d^3\vec k_i}{(2\pi)^3 2E_i}\mathcal{M}_{fm}\mathcal{M}^*_{im}(2\pi)^4\delta^4(p_f-p_m)(2\pi)^4\delta^4(p_f-p_i).$$

Take $f=i$, get

$$2Im\mathcal{M}_{ii}=\sum_m\int d\Pi_m|\mathcal{M}_{im}|^2,$$

where $d\Pi_m$ is the density of states for the process $i\to m$. This is the optical theorem. It relates the imaginary part of the forward scattering amplitude to the total cross section, e.g.

$$\mathrm{Im}\mathcal{M}(\mathrm{k_1,k_2\to k_1,k_2})=2\mathrm{E_{cm}p_{cm}}\sigma_{\mathrm{tot}}(\mathrm{k_1,k_2\to anything}).$$

Recall that the imaginary part of amplitudes is discontinuous across the cut starting at $s = 4m^2$. So we can there relate

$$Disc\mathcal{M}(s) = 2iIm\mathcal{M}(s) \sim \int d\Pi \left|\mathcal{M}_{cih}\right|^2 \sim \sigma_{tot}$$

where cih means cut in half.

Consider e.g. the 1-loop contribution to the 4-point amplitude in $\lambda\phi^4$, in the s channel

$$\mathcal{M}^{(1)} = \tfrac{1}{2}\lambda^2 \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(\frac{1}{2}p+k)^2 - m^2 + i\epsilon} \frac{1}{(\frac{1}{2}p-k)^2 - m^2 + i\epsilon},$$

where $p = p_1 + p_2$. Recall that we evaluated this as (with $s = p^2$)

$$\frac{\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\frac{4\pi\mu^2}{m^2} + A(s),\right)$$

where

$$A(s) = 2 - \sqrt{1 - 4m^2/s}\log\left(\frac{\sqrt{1-4m^2/s}+1}{\sqrt{1-4m^2/s}-1}\right).$$

The $1/\epsilon$ term (together with some constants, depending on our scheme) is cancelled by a counterterm diagram. The function $A(s)$ remains. The threshold is where $s = 4m^2$. Below threshold, the amplitude is purely real. Above threshold, there is a discontinuous imaginary part, with

$$Disc\mathcal{M}(s) = 2iIm\mathcal{M}(s) \sim \int d\Pi \left|\mathcal{M}_{cih}\right|^2 \sim \sigma_{tot}$$

where cih means cut in half. The tree-level scattering amplitude comes from the imaginary part of the one-loop amplitude.

- Let's go back to

$$\tilde{\Gamma}_B^{(n)}(p_1, \dots p_n; \lambda_B, m_B, \epsilon) = Z_\phi^{-n/2}\tilde{\Gamma}_R^{(n)}(p_1, \dots p_n; \lambda_R, m_R, \mu, \epsilon).$$

For fixed physics, the LHS is some fixed quantity. The RHS depends on the renormalization point μ and the scheme. The LHS does not! This leads to what is known as the renormalization group equations, which state how the renormalized quantities must vary with μ.

Take $d/d\ln\mu$ of both sides, and use $d\Gamma_B/d\mu = 0$. This gives

$$\left(\frac{\partial}{\partial\ln\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} + \gamma_m m_R\frac{\partial}{\partial\ln m_R} - n\gamma\right)\tilde{\Gamma}_R^{(n)}(p_1, \dots p_n; \lambda_R, m_R, \mu) = 0$$

Here

$$\beta(\lambda) \equiv \frac{d}{d \ln \mu} \lambda_R$$

$$\gamma = \tfrac{1}{2} \frac{d}{d \ln \mu} \ln Z_\phi$$

$$\gamma_m = \frac{d \ln m_R}{d \ln \mu}.$$

This is the Callan-Symanzik equation. It can be integrated, to relate the renormalized Greens functions at different scales μ and μ'. Let us focus on what β and γ mean.

• Understand what β and γ mean: the bare quantities are some function of the renormalized ones and epsilon. E.g. for $\lambda\phi^4$ in MS we have

$$\lambda_B = \mu^\epsilon(\lambda + \delta_\lambda) \equiv \mu^\epsilon \lambda Z_\lambda$$

Let us write

$$Z_\lambda \equiv 1 + \sum_k a_k(\lambda)\epsilon^{-k},$$

where we found $a_1(\lambda) = +3\lambda/16\pi^2$ to one loop. The bare parameter λ_B is independent of μ, whereas λ depends on μ, such that the above relation holds. Take $d/d \ln \mu$ of both sides,

$$0 = \epsilon \lambda Z_\lambda + \beta(\lambda, \epsilon) Z_\lambda + \beta(\lambda, \epsilon)\lambda \frac{dZ_\lambda}{d\lambda}.$$

Using the above expansion for Z_λ and requiring that $\beta(\lambda, \epsilon)$ be regular at $\epsilon = 0$, so $\beta(\lambda, \epsilon) = \beta(\lambda) + \sum_n \beta_n \epsilon^n$, gives

$$\beta(\lambda, \epsilon) = -\epsilon\lambda + \beta(\lambda)$$

$$\beta(\lambda) = \lambda^2 \frac{da_1}{d\lambda}$$

$$\lambda^2 \frac{da_{k+1}}{d\lambda} = \beta(\lambda) \frac{d}{d\lambda}(\lambda a_k).$$

The beta function is determined entirely from a_1. The $a_{k>1}$ are also entirely determined by a_1. In k-th order in perturbation theory, the leading pole goes like $1/\epsilon^k$.

We find for $\lambda\phi^4$

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3).$$

Integrating, this gives

$$\lambda = \lambda_0 \left(1 - \frac{3}{16\pi^3} \lambda_0 \ln(\mu/\mu_0)\right)^{-1}.$$

We similarly have $m_B^2 = Z_m m_R^2$ and

$$\gamma_m(\lambda) = \tfrac{1}{2}\lambda \frac{dZ_m^{(1)}}{d\lambda} = \tfrac{1}{2}\frac{\lambda}{16\pi^2} - \frac{5}{12}\frac{\lambda^2}{6(16\pi^2)^2} + \dots$$

where $Z_m^{(1)}$ means the coefficient of $1/\epsilon$ and $\dots$ are higher orders in perturbation theory, and

$$\gamma_\phi = -\tfrac{1}{2}\lambda \frac{d}{d\lambda} Z_\phi^{(1)} = \frac{1}{12}\frac{\lambda^2}{(16\pi^2)^2} + \dots$$

For any gauge invariant field ϕ, we always have $\gamma_\phi \geq 0$, where $\gamma_\phi = 0$ iff it is a free field. This follows from the spectral decomposition result that $Z \leq 1$.

• Note: $\beta > 0$ means the coupling is small in the IR, and large in the UV. Such theories are "not asymptotically free" or are "IR free." Most theories are like this, e.g. $\lambda\phi^4$, QED, Yukawa interactions. If $\beta < 0$, then the coupling is small in the UV, and large in the IR. Such theories are "asymptotically free;" only non-Abelian gauge theories, like QCD, are like that.