1/8/07 Lecture 2 outline

• Computation. Integral can be broken into time slices, as way to define it. E.g. free particle

$$\left(\frac{-im}{2\pi\hbar\epsilon}\right)^{N/2}\int\prod_{i=1}^{N-1}dx_i\exp[\frac{im}{2\hbar\epsilon}\sum_{i=1}^{N}(x_i-x_{i-1})^2]$$

Where we take $\epsilon\to 0$ and $N\to\infty$, with $N\epsilon=T$ held fixed.

Do integral in steps. Apply expression for real gaussian integral (valid: analytic continuation). After $n-1$ steps, get integral:

$$\left(\frac{2\pi i\hbar n\epsilon}{m}\right)^{-1/2}\exp[\frac{m}{2\pi i\hbar n\epsilon}(x_n-x_0)^2].$$

So the final answer is

$$U(x_b,x_a;T)=\left(\frac{2\pi i\hbar T}{m}\right)^{-1/2}\exp[im(x_b-x_a)^2/2\hbar T].$$

Note that the exponent is $e^{iS_{cl}/\hbar}$, where S_{cl} is the classical action for the classical path with these boundary conditions. (More generally, get a similar factor of $e^{iS_{cl}/\hbar}$ for interacting theories, from evaluating path integral using stationary phase.)

Plot phase of Uas a function of $x=x_b-x_a$, fixed T, Lots of oscillates. For large x, nearly constant wavelength λ, with

$$2\pi=\frac{m(x+\lambda)^2}{2\hbar T}-\frac{2m^2}{2\hbar T}\approx\frac{mx\lambda}{\hbar T}=p\lambda/\hbar.$$

Gives $p=\hbar k$! Recover $\psi\sim e^{ipx/\hbar}$. More generally, get $p=\hbar^{-1}k$, with $p=\partial S_{cl}/\partial x_b$ (can show $p=\partial L/\partial\dot{x}=\partial S_{cl}/\partial x_b$. Can also recover $\psi\sim e^{-i\omega T}$, with $\omega=\hbar^{-1}(-\partial S_{cl}/\partial t_b)$. Agrees with $E=\hbar\omega$, since $E=p\dot{x}-L=-\partial S_{cl}/\partial t_b$.

• Nice application: Aharonov-Bohm. Recall $L=\frac{1}{2}m\dot{\vec{x}}^2+q\dot{\vec{x}}\cdot\vec{A}-q\phi$. Solenoid with $B\neq 0$ inside, and $B=0$ outside. Phase difference in wavefunctions is

$$e^{i\Delta S/\hbar}=e^{iq\oint\vec{A}\cot d\vec{x}/\hbar}=e^{iq\Phi/\hbar}.$$

Aside on Dirac quantization for magnetic monopoles.

• The nice thing about the path integral is that it generalizes immediately to quantum fields, and for that matter to all types (scalars, fermions, gauge fields). Consider first scalars fields

- Compute Green's functions via

$$\langle 0|T\prod_{i=1}^{n}\phi_H(x_i)|0\rangle/\langle 0|0\rangle = Z_0^{-1}\int[d\phi]\prod_{i=1}^{n}\phi(x_i)\exp(iS/\hbar),$$

with $Z_0=\int[d\phi]\exp(iS/\hbar)$.

Ordinary (non functional), multi-dimensional gaussian integrals:

$$\prod_{i=1}^{N}d\phi_i\exp(-(\phi,B\phi))=\pi^{N/2}(\det B)^{-1/2},$$

where $(\phi,B\phi)=\sum_i\phi_i(B\phi)_i$ and $(B\phi)_i=\sum_j B_{ij}\phi_j$. The integral was evaluated by changing variables in the $d\phi_i$, to the eigenvectors of the symmetric matrix B; then the integrals decouple into a product of simple 1-variable gaussians.

By analogy, consider Z_0 for a free Klein Gordon field:

$$Z_0=\int[d\phi]e^{iS/\hbar}\qquad S=\tfrac{1}{2}\int d^4x\phi(x)(-\partial^2-m^2)\phi(x),$$

where we integrated by parts and dropped a surface term. The analogy with the above has $B\sim-\partial^2-m^2$, and $(\phi,B\phi)\sim S$, so

$$Z_0=\mathrm{const}(\det(-\partial^2-\mathrm{m}^2))^{-1/2}.$$

We will have to explain how to handle the functional determinant, $\det(-\partial^2-m^2)$