

1/24/07 Lecture 5 outline

- We got the functional integral to converge via the  $i\epsilon$ . There is another way, which is often very useful: Wick rotate to Euclidean space. The  $k_0$  momentum integral, like the one above, can be analytically continued, as long as no poles are crossed. This allows continuation to  $k_0 = ik_4$ , with  $k_4$  real. Then  $k^2 = -k_E^2$ , and  $d^4k = id^4k_E$ . To avoid having  $e^{ikx}$  blow up anywhere, we also continue time:  $x_0 = -ix_4$ , so  $d^4x = -id^4x_E$ . The Feynman propagator, in Euclidean space, is

$$\Delta_E(x) = \int \frac{d^4k_E}{(2\pi)^4} e^{-ikx} \frac{1}{k_E^2 + m^2},$$

where we can now drop the  $i\epsilon$ , since it's no longer needed. Note  $k_E^2 + m^2$  is never zero, so the integrand never has a pole, and the solution  $\Delta_E$  is unique.

The action changes as  $S = \int d^4x (\frac{1}{2} \partial\phi\partial\phi - V) = i \int d^4x_E (\frac{1}{2} \partial_{x_E}\phi\partial_{x_E}\phi + V) = iS_E$ , where  $S_E$  looks like the energy now,  $S_E = "H"$ ! Then

$$\int [d\phi] \exp[\frac{i}{\hbar} S] \rightarrow \int [d\phi] e^{-\frac{1}{\hbar} "H"}$$

which is like the partition function of stat mech (as you saw in your HW)! (But here " $H$ " is like the Hamiltonian of a theory living in 4 spatial dimensions..). Note  $\hbar$  here appears as does  $T$  (temperaure) there, connects intuition of quantum fluctuations with intuition of thermal fluctuations!

It is sometimes useful to do all Feynman diagram computations in Euclidean space, and analytically continue back to Minkowski at the end of the day. The Euclidean space Feynman rules work like this:  $(k_E^2 + m^2)^{-1}$  for each propagator,  $d^4k_E$  for each loop, and  $-g$  for each vertex. Comparing with what we had before, we have dropped some factors of  $i$ :

$$i^{L+V-I} = i,$$

since (connected) diagrams have  $L = I - V + 1$ . So every diagram in the sum just differs by a factor of  $i$ , so the sums work the same as before (no relative differences).

- For computing S-matrix elements, we will especially be interested in *connected* Green's functions. There are nice combinatoric formulae (you might have already seen some last quarter?). E.g.

$$\sum \text{all diagrams} = \left( \sum \text{"connected"} \right) \cdot \exp\left( \sum \text{disconnected vacuum bubbles} \right).$$

And the vacuum bubble diagrams cancel. We write “connected” because for  $n > 2$  point functions there are still disconnected diagrams, connected to the external points, included in this sum. But even those disconnected diagrams drop out when we consider  $S - 1$ : they correspond to the 1. In the end, we’re interested in the fully connected diagrams. It turns out that there is a generating functional for them. (N.B. sometimes people reverse the names of what I’m calling  $W$  and  $Z$ !. Peskin calls  $W \rightarrow E$ .) Defining,

$$iW[J] \equiv \ln Z[J]$$

$iW[J]$  is the generating functional for connected Green’s functions

$$G_{conn}^{(n)}(x_1, \dots, x_n) = \hbar^{-1} \prod_{j=1}^n \frac{-i\delta}{\delta J(x_j)} iW[J],$$

i.e.

$$iW[J] = \hbar \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n G_{conn}^{(n)}(x_1, \dots, x_n) J(x_1) \dots J(x_n).$$

In momentum space, we can write:

$$iW[J] = \hbar \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \dots \frac{d^4k_n}{(2\pi)^4} \tilde{J}(-k_1) \dots \tilde{J}(-k_n) \tilde{G}_c(k_1, \dots, k_n).$$

- Examples, to illustrate how  $iW[J] \equiv \ln Z[J]$  gives the connected diagrams. First

$$-i \frac{\delta iW}{\delta J} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(x)} = \langle \phi(x) \rangle_J.$$

$$(-i)^2 \frac{\delta^2}{\delta J(x) \delta J(y)} (iW) = \langle \phi(x) \phi(y) \rangle_J - \langle \phi(x) \rangle_J \langle \phi(y) \rangle_J.$$

Note that  $\langle \phi(x) \phi(y) \rangle$  has two types of contributions, connected and disconnected; the 2nd term precisely cancels off the disconnected ones. Similarly  $\delta W / \delta J^3$  has terms like  $\langle \phi \phi \phi \rangle - \langle \phi \phi \rangle \langle \phi \rangle + 2 \langle \phi \rangle \langle \phi \rangle \langle \phi \rangle$ , which give precisely  $\langle \phi \phi \phi \rangle_{connected}$ . Can prove by induction that the *log* in  $W$  properly subtracts away all non-connected diagrams!

- Will later discuss LSZ: how to relate Green’s functions to S-matrix elements (and hence physical observables). Will see there that only connected diagrams contribute; this is why  $W$  is useful.