Homework 1, Due Jan. 17, 2008

1. Recall the QM discussion. The position is ϕ^i, with $i = 1 \dots D$. The momentum is Π_j. Let $\mathcal{O}$ be an arbitrary function of these variables. Suppose operator A acts on $\mathcal{O}$ as $[A, \mathcal{O}] = -A^d \mathcal{O}$, where A^d is a differential operator (derivatives w.r.t. ϕ^i and Π_j). Suppose that there are operators B and C, with $[B, \mathcal{O}] = -B^d \mathcal{O}$, and $[C, \mathcal{O}] = -C^d \mathcal{O}$. Show that $[A, B] = C$ implies $[A^d, B^d] = C^d$, but that this would not have worked without the minus signs. (As a concrete realization of this, you can think of $[L_x, L_y] = i\hbar L_z$, with $L_z^d = i\hbar \frac{\partial}{\partial \phi}$ and similar standard expressions for L_x^d and L_y^d.)

2. Recall $N = 2$ SQM from class, with

$$\begin{aligned}
\theta &= \frac{1}{\sqrt{2}}(\theta^1 + i\theta^2) \quad ; \quad \bar{\theta} = \frac{1}{\sqrt{2}}(\theta^1 - i\theta^2) \\
D &= \frac{1}{\sqrt{2}}(D_1 - iD_2) = \partial_\theta - i\bar{\theta}\partial_t \quad ; \quad \bar{D} = \frac{1}{\sqrt{2}}(D_1 + iD_2) = \partial_{\bar{\theta}} - i\theta\partial_t \\
Q^d &= \frac{1}{\sqrt{2}}(Q_1^d - iQ_2^d) = \partial_\theta + i\bar{\theta}\partial_t \quad ; \quad \bar{Q}^d = \frac{1}{\sqrt{2}}(Q_1^d + iQ_2^d) = \partial_{\bar{\theta}} + i\theta\partial_t
\end{aligned} \tag{1}$$

Consider the QM lagrangian

$$L = \int d^2\theta \left(\tfrac{1}{2} D\Phi \bar{D}\Phi + W(\Phi) \right).$$

Here $d^2\theta \equiv d\bar{\theta} d\theta$, so $\int d^2\theta \theta\bar{\theta} = +1$. Take Φ to be a real superfield

$$\Phi = \phi + i\theta\psi + i\bar{\theta}\psi^\dagger + \theta\bar{\theta}F.$$

(a) Compute the Lagrangian in components.

(b) Notice that ϕ and ψ have usual looking kinetic terms, but F has no kinetic term. Such non-dynamical **auxiliary fields**, like F, are needed for implementing susy off shell. On shell, they can simply be eliminated by their equations of motion: $\partial L / \partial F = 0$. Show that, upon eliminating the F, the potential for the field ϕ is $V(\phi) = \frac{1}{2}(W')^2$.

3. Now consider the QM lagrangian

$$\mathcal{L} = \int d^2\theta \frac{1}{4} \mathcal{F}\mathcal{F}^\dagger + \int d\theta W(\Phi)\mathcal{F} + \int d\bar{\theta} \bar{W}(\Phi^\dagger)\mathcal{F}^\dagger,$$

where Φ and $\mathcal{F}$ are chiral $N=2$ scalar and fermionic superfields, so $\bar{D}\Phi = \bar{D}\mathcal{F} = 0$. Recall from class that chiral superfields can be written as

$$\begin{aligned}\Phi(y,\theta) &= \phi(y) + \sqrt{2}\theta\psi(y) = \phi(t) + \sqrt{2}\theta\psi(t) - i\theta\bar{\theta}\dot{\phi}(t)\\ \mathcal{F}(y,\theta) &= \chi(y) + \sqrt{2}\theta F(y) = \chi(t) + \sqrt{2}\theta F(t) - i\theta\bar{\theta}\dot{\chi}\end{aligned}$$

where ϕ and ψ and F are complex, and where $y \equiv t - i\theta\bar{\theta}$. The difference between Φ and $\mathcal{F}$ is that the first component ϕ is bosonic, whereas χ is fermionic. The $\mathcal{F}^\dagger$ and $\Phi^\dagger$ are anti-chiral, i.e. annihilated by D, and can be expanded as

$$\begin{aligned}&\Phi^\dagger(\bar{y},\bar{\theta}) = \phi^\dagger(\bar{y}) - \sqrt{2}\bar{\theta}\psi^\dagger(\bar{y}) = \phi^\dagger(t) - \sqrt{2}\bar{\theta}\psi^\dagger(t) + i\theta\bar{\theta}\dot{\phi}^\dagger(t)\\ &\mathcal{F}^\dagger(\bar{y},\bar{\theta}) = \chi^\dagger(\bar{y}) + \sqrt{2}\bar{\theta}F^\dagger(\bar{y}) = \chi^\dagger(t) + \sqrt{2}\bar{\theta}F^\dagger(t) + i\theta\bar{\theta}\dot{\chi}^\dagger(t)\\ &\bar{y} \equiv t + i\theta\bar{\theta}\end{aligned}$$

(a) Using the fact that Q^d and $\bar{Q}^d$ generate the supersymmetry transformations, verify that the above Lagrangian is invariant under susy for any *holomorphic* function $W(\Phi)$, i.e. it must satisfy the Cauchy-Riemann equations $\frac{\partial W(\Phi)}{\partial \Phi^\dagger} = 0$.

(b) Write out $\mathcal{L}$ in components. Eliminate the auxiliary fields, and verify that the scalar potential satisfies $V \geq 0$.

3. (From Argyres.) Show that conservation of a symmetric, traceless charge $Q^{\mu\nu}$, together with energy momentum conservation implies

$$p_1^\mu p_1^\nu + p_2^\mu p_2^\nu = q_1^\mu q_1^\nu + q_2^\mu q_2^\nu$$

for elastic scattering of two identical scalars with incoming momenta p_1 and p_2 and outgoing momenta q_1 and q_2. Show that this implies that the scattering angle is zero.

4. Argyres susy 1996, exercise 2.2.

5. Argyres (1996) exercise 3.3.