1/5/09 Lecture outline

• We'll focus for a while on scalar field theory, e.g. $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$, with e.g. $V(\phi) = \frac{1}{2}m^2\phi^2 + V_{int}(\phi)$, with e.g. $V_{int}(\phi) = \frac{1}{4!}\lambda\phi^4$.

• Last quarter you learned canonical quantization. The field $\phi(x)$ is analogous to $q(t)$ in QM (indeed, QM is field theory in $d = 1$ dimension), and its conjugate momentum is $\Pi = \partial\mathcal{L}/\partial\dot{\phi}$. These are operators, with equal time commutators

$$[\phi(t,\vec{x}), \Pi(t,\vec{x}')] = i\hbar\delta^3(\vec{x} - \vec{x}').$$

We'll usually set $\hbar = 1$, aside from occasionally making it explicit to emphasize some physics. It's often more convenient to work in momentum space,

$$\phi(x) = \int \frac{d^3k}{(2\pi)^2 2\omega}[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}]$$

and then the canonical quantization rules imply that

$$[a(k), a^\dagger(k')] = (2\pi)^2 2\omega\delta^3(\vec{k} - \vec{k}'),$$

with others vanishing. Creation and annihilation operators, act on $|0\rangle$, with $a(\vec{k})|0\rangle = 0$ and $a^\dagger(k)|\rangle = |k\rangle$. Can create a packet with momentum localized around $\vec{k}_1$ via $a_1^\dagger|0\rangle$, with $a_1^\dagger = \int d^3k f_1(k)a^\dagger(k)$ and e.g. $f_1(k)$ a gaussian centered around $\vec{k}_1$. • Consider scattering n incoming particles into n' outgoing ones: $|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)\ldots a_n^\dagger(t)|0\rangle$ and $|f\rangle = \lim_{t\to+\infty} a_{1'}^\dagger(t)\ldots a_{n'}^\dagger(t)|0\rangle$. The scattering amplitude is the S-matrix element $\langle f|S|i\langle$. Taking the initial and final states to differ, we'll just write this as

$$\langle f|i\rangle = \langle 0|Ta_{1'}\ldots a_{n'}a_1^\dagger\ldots a_n^\dagger|0\rangle.$$

Now use

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{\infty} dt\partial_0 a_1^\dagger(t) = -i\int d^3\vec{k}f_1(\vec{k})\int d^4x e^{-ikx}(-\partial^2 + m^2)\phi(x),$$

where the steps needed for the 2nd equality can be found in full detail in e.g. Srednicki. Using this we obtain

$$\begin{aligned}\langle f|i\rangle &= \langle k_{1'}\ldots k_{n'}|k_1\ldots k_n\rangle \\ &= i^{n+n'}\prod_{j'=1}^{n'}\int d^4x_j' e^{ik_j'x_j'}(\partial_{j'}^2 + m^2)\prod_{j=1}^{n} e^{-ik_jx_j}(\partial_j^2 + m^2)G_{n+n'}(x_1\ldots x_n, x_{1'}\ldots x_{n'}),\end{aligned} \tag{1}$$

where

$$G_{n+n'}(x_1 \dots x_n, x_{1'} \dots x_{n'}) \equiv \langle 0|T\phi(x_{1'}) \dots \phi(x_{n'})\phi(x_1) \dots \phi(x_n)|0\rangle. \quad (2)$$

This is the LSZ formula, which we'll discuss further later. So to compute scattering amplitudes, and thus cross sections and decay rates etc, we just need to compute the above Green's functions, involving time ordered products of fields.

• This is also expressed as Dyson's formula, and recall also Wick's theorem:

$$U(t_2, t_1) = T \exp(-i \int_{t_1}^{t_2} H_{int}(t')dt'),$$

$$T(\phi_1 \dots \phi_n) =: \phi_1 \dots \phi_n : + : \text{all contractions} :$$

This is nicely expressed in terms of Feynman diagams.

• Our first topic is the Feynman path integral. Gives another way to quantize particles, and fields. For particles, consider time evolution operator

$$U(x_a, x_b; T) = \langle x_b|e^{-iHT/\hbar}|x_a\rangle.$$

Satisfies SE

$$i\hbar\partial_T U = HU.$$

Feynman:

$$U(x_a, x_b; T) = \int [dx(t)]e^{-S[x(t)]/\hbar}.$$

Integral can be broken into time slices, as way to define it. Discuss how to derive this formula from the usual description of QM with operators, by introducing the time slices and a complete set of q and p eigenstates at each step.

E.g. free particle

$$\left(\frac{-im}{2\pi\hbar\epsilon}\right)^{N/2} \int \prod_{i=1}^{N-1} dx_i \exp[\frac{im}{2\hbar\epsilon}\sum_{i=1}^{N}(x_i - x_{i-1})^2]$$

Where we take $\epsilon \to 0$ and $N \to \infty$, with $N\epsilon = T$ held fixed.

Do integral in steps. After $n-1$ steps, get integral

$$(\frac{2\pi i\hbar n\epsilon}{m})^{-1/2} \exp[\frac{m}{2i\hbar n\epsilon}(x_n - x_0)^2].$$

So final answer is

$$U(b, a) = [\frac{2\pi i\hbar T}{m}]^{-1/2} \exp[im(x_b - x_a)^2/2\hbar T].$$

Plot this as a function of $x = (x_b - x_a)$: lots of oscillations. For large x, nearly constant wavelength λ, where

$$2\pi = \frac{m(x+\lambda)^2}{2\hbar T} - \frac{mx^2}{2\hbar t} \approx \frac{mx\lambda}{\hbar T}.$$

Gives $p = \hbar k$! More generally, get $k = \hbar^{-1} p$ with $p = \partial S / \partial x_b$. Can show $p = \partial L / \partial \dot{x} = \partial S_{cl} / \partial x_b$. So recover $\psi \sim e^{ipx/\hbar}$. Can also recover $\psi \sim e^{-i\omega T}$, with $\omega = -\hbar^{-1} \partial S_{cl} / \partial t_b$. Agrees with $E = \hbar\omega$, since $E = -\partial S_{cl} / \partial t_b$.

Generalization to quantum field theory is immediate:

$$\langle \phi_b(\vec{x}, T) | e^{-iHT} | \phi_a(\vec{x}, 0) \rangle = \int [d\phi] e^{iS/\hbar} \qquad S = \int d^4x \mathcal{L}.$$

This is then used to compute Green's functions:

$$\langle \Omega | T \prod_{i=1}^{n} \phi_H(x_i) | \Omega \rangle = Z_0^{-1} \int [d\phi] \prod_{i=1}^{n} \phi(x_i) \exp(iS/\hbar),$$

with $Z_0 = \int [d\phi] \exp(iS/\hbar)$.