2/11/10 Lecture 11 outline

• The quantity appearing in our functional integral is S_B. We split $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{c.t.}$. For fixed physics, the LHS is some fixed quantity. How we split it up on the RHS depends on our renormalization scheme.

• Last time: $\lambda\phi^4$. Recall

$$\mathcal{L}_{c.t.} = \tfrac{1}{2}(Z_\phi - 1)\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}(m_B^2 Z_\phi - m^2)\phi^2 - (\lambda_B Z_\phi^2 - \lambda\mu^\epsilon)\frac{1}{4!}\phi^4.$$

Define $\delta_Z \equiv Z_\phi - 1$, $\delta_m = m_B^2 Z_\phi - m^2$, $\delta_\lambda\mu^\epsilon = \lambda_B Z_\phi^2 - \lambda\mu^\epsilon$. Last time we considered the "on shell" renormalization scheme, defined by imposing

$$\Pi'(m^2) = 0, \qquad \frac{d\Pi'}{dp^2}|_{p^2=m^2} = 0, \qquad \tilde{\Gamma}^{(4)}|_{s=\mu} = -\lambda$$

Last time we took $\mu = 4m^2$. We could also change the renormalization point μ.

• Now mention two other renormalization schemes, which have an advantage in actual perturbative calculations in that they are *mass independent* (to be illustrated below). In minimal subtraction (MS) we choose the counterterms to remove the $1/\epsilon$ poles, and nothing else. A variant is $\overline{MS}$, where one replaces

$$\frac{\Gamma(2-\frac{1}{2}D)}{(4\pi)^{D/2}(m^2)^{2-\frac{1}{2}D}} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \log(4\pi/m^2)\right)$$

with

$$\frac{1}{16\pi^2}\log(M^2/m^2),$$

for some arbitrary mass parameter M. (The advantage is that it gets rid of annoying finite constants like γ and other derivatives of the gamma function, which otherwise proliferate at each higher loop order.) The apparent freedom to define things many different ways always cancels out at the end of the day, when one relates to physical observables. Different choices have different benefits along the way.

• Let's consider $\lambda\phi^4$ in MS. Recall that we have

To one loop, we have

$$\delta_m = \frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_\lambda = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_Z = 0.$$

Now consider the propagator to two loops. Diagram 1 is a one-loop diagram with the 1-loop $\delta\lambda$ counterterm at the vertex. Diagram 2 is a one-loop diagram with the 1-loop δ_m

counterterm on the internal propagator. Diagram 3 is a two-loop diagram which looks like a double-scoop of the 1-loop diagrams. Diagram 4 is the one from your HW. Diagram 5 have no loops, but an insertion of the 2-loop δ_m and δ_Z counter terms. Let's consider the pole terms in the diagrams. Diagram 1 gives

$$i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{3}{2}\left(\frac{2}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{m^2}{4\pi\mu^2}+\frac{1}{\epsilon}-\frac{\gamma}{\epsilon}\right)+\mathcal{O}(\epsilon^0)$$

Diagram 2 gives

$$i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{1}{2}\left(\frac{2}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{m^2}{4\pi\mu^2}-\frac{\gamma}{\epsilon}\right)+\mathcal{O}(\epsilon^0)$$

Diagram 3 gives

$$-i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{1}{2}\left(\frac{2}{\epsilon^2}-\frac{2}{\epsilon}\ln\frac{m^2}{4\pi\mu^2}+\frac{1}{\epsilon}-\frac{2\gamma}{\epsilon}\right)+\mathcal{O}(\epsilon^0)$$

Diagram 4 gives

$$i\frac{\lambda^2}{(16\pi^2)^2}\left(-\frac{m^2}{\epsilon^2}+\frac{1}{\epsilon}\left(m^2\ln\frac{m^2}{4\pi\mu^2}+\frac{1}{12}p^2+(\gamma-\frac{3}{2}m^2)\right)\right)$$

Diagram 5 are the two-loop counterterms, $i\delta_Z^{(2)}p^2-i\delta_m^{(2)}$. We should then take for the 2-loop contributions to the counterterms

$$\delta m^{(2)}=\frac{\lambda^2}{(16\pi^2)^2}\left(\frac{2}{\epsilon^2}-\frac{1}{2\epsilon}\right)m^2,$$

$$\delta_Z^{(2)}=-\frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{\epsilon}.$$

The terms involving $\ln m^2/4\pi\mu^2$ all cancel. This happens for all loops. MS is a mass independent scheme, in that $\delta\lambda$, δZ, and $\delta m/m^2$ are independent of m and μ.

• Renormalizability: all divergences cancelled by counter terms of the same form as original $\mathcal{L}$. This would not be the case for e.g. $\lambda\phi^6$. Even for $\lambda\phi^4$, it is quite non-trivial. For example, in doing 2 loops, there could have been some term from one loop diagrams, with counter terms, leading to $\frac{1}{\epsilon}\ln p^2$, which could not be cancelled by a counterterm in our lagrangian. Sometimes individual diagrams indeed behave like that. But the coefficients of all such terms sum to zero.

• Renormalized and bare Greens functions. Recall that $\Phi_B\equiv Z_\phi^{1/2}\phi$, and the definition of the 1PI Green's functions $\widetilde{\Gamma}^{(n)}$, and in particular that they have all n external propagators amputated. It then follows that

$$\tilde{\Gamma}_B^{(n)}(p_1,\ldots p_n;\lambda_B,m_B,\epsilon)=Z_\phi^{-n/2}\tilde{\Gamma}_R^{(n)}(p_1,\ldots p_n;\lambda_R,m_R,\mu,\epsilon).$$

For fixed physics, the LHS is some fixed quantity. The RHS depends on the renormalization point μ and the scheme. The LHS does not! This leads to what is known as the renormalization group equations, which state how the renormalized quantities must vary with μ. Rewrite above as

$$Z_\phi^{n/2}\tilde{\Gamma}_B^{(n)}(p_1,\ldots p_n;\lambda_B,m_B,\epsilon)=\tilde{\Gamma}_R^{(n)}(p_1,\ldots p_n;\lambda_R,m_R,\mu,\epsilon).$$

Now the RHS is finite, so the LHS must be too. So we can take $\epsilon\to 0$ without a problem.

Will develop this next time, after a little detour on Z and related things.