

2/17/10 Lecture 13 outline

- Last time:

$$-i\tilde{\Delta}(p^2) = \frac{i}{p^2 - m^2 - \Pi'(p^2) + i\epsilon} = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{\sim 4m^2}^{\infty} \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon}.$$

So, using $\frac{1}{x \pm i\epsilon} = P(1/x) \mp i\pi\delta(x)$, argue that $\pi\rho(s) = 2Im\tilde{\Delta}(s)$ for $s \geq 4m^2$. (The minus sign in the definition of $\tilde{\Delta}$ above is related to the special definition of $\tilde{\Gamma}^{(n)}$ for $n = 2$ and $\tilde{\Delta} \sim 1/\tilde{\Gamma}^{(2)}$.)

- Example from last quarter: tree-level contribution to the Compton effect, scattering light off an electron. The S matrix element is given at tree-level by $S = 1 + iT$, where

$$\langle f | iT | i \rangle = i(2\pi)^4 \delta^4(k_f + p_f - k_i + p_i) \mathcal{M}_{fi}$$

$$\mathcal{M}_{fi} = -e^2 \bar{u}(p_f, \alpha_f) \left(\not{\epsilon}_f \frac{1}{\not{p}_i + \not{k}_i - m} \not{\epsilon}_i + \not{\epsilon}_i \frac{1}{\not{p}_i - \not{k}_f - m} \not{\epsilon}_f \right) u(p_i, \alpha_i).$$

More generally, the S-matrix element is given according to LSZ by the connected, amputated Greens functions. Note that it is not just the 1PI diagrams contributing (the above example is a non-1PI contribution).

- Analyticity properties. E.g. $2 \rightarrow 2$ scattering. $\mathcal{M}(s) = \mathcal{M}(s^*)^*$. The real part $Re\mathcal{M}$ is continuous across the real axis, whereas the Im part picks up a minus sign. So the discontinuity $Disc\mathcal{M}(s) = 2iIm\mathcal{M}(s + i\epsilon)$. E.g. $\frac{1}{x \pm i\epsilon} = P(1/x) \mp i\pi\delta(x)$ shows that the discontinuity of $\frac{1}{p^2 - m^2 + i\epsilon}$ is $-2\pi i\delta(p^2 - m^2)$.

- Optical theorem. The S-matrix $S = U(t_f = \infty, t_i = -\infty)$ is unitary, $S^\dagger S = 1$. Write $S = 1 + iT$, then get $2Im(T) \equiv -i(T - T^\dagger) = T^\dagger T$. Thus

$$-i(2\pi)^4 \delta^4(p_f - p_i) (\mathcal{M}_{fi} - \mathcal{M}_{if}^*) = \sum_m \prod_j \int \frac{d^3 \vec{k}_j}{(2\pi)^3 2E_j} \mathcal{M}_{fm} \mathcal{M}_{im}^* (2\pi)^4 \delta^4(p_f - p_m) (2\pi)^4 \delta^4(p_f - p_i).$$

Take $f = i$, get

$$2Im\mathcal{M}_{ii} = \sum_m \int d\Pi_m |\mathcal{M}_{im}|^2,$$

where $d\Pi_m$ is the density of states for the process $i \rightarrow m$. This is the optical theorem. It relates the imaginary part of the forward scattering amplitude to the total cross section, e.g.

$$Im\mathcal{M}(k_1, k_2 \rightarrow k_1, k_2) = 2E_{cm} p_{cm} \sigma_{tot}(k_1, k_2 \rightarrow \text{anything}).$$

Recall that the imaginary part of amplitudes is discontinuous across the cut starting at $s = 4m^2$. So we can there relate

$$Disc\mathcal{M}(s) = 2iIm\mathcal{M}(s) \sim \int d\Pi |\mathcal{M}_{cjh}|^2 \sim \sigma_{tot}$$

where *cjh* means cut in half.

Consider e.g. the 1-loop contribution to the 4-point amplitude in $\lambda\phi^4$, in the s channel

$$\mathcal{M}^{(1)} = \frac{1}{2}\lambda^2 \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(\frac{1}{2}p+k)^2 - m^2 + i\epsilon} \frac{1}{(\frac{1}{2}p-k)^2 - m^2 + i\epsilon},$$

where $p = p_1 + p_2$. Recall that we evaluated this as (with $s = p^2$)

$$\frac{\lambda^2}{32\pi^2} \left(\frac{2}{\epsilon} - \gamma + \log \frac{4\pi\mu^2}{m^2} + A(s), \right)$$

where

$$A(s) = 2 - \sqrt{1 - 4m^2/s} \log \left(\frac{\sqrt{1 - 4m^2/s} + 1}{\sqrt{1 - 4m^2/s} - 1} \right).$$

The $1/\epsilon$ term (together with some constants, depending on our scheme) is cancelled by a counterterm diagram. The function $A(s)$ remains. The threshold is where $s = 4m^2$. Below threshold, the amplitude is purely real. Above threshold, there is a discontinuous imaginary part, with

$$Disc\mathcal{M}(s) = 2iIm\mathcal{M}(s) \sim \int d\Pi |\mathcal{M}_{cjh}|^2 \sim \sigma_{tot}$$

where *cjh* means cut in half. The tree-level scattering amplitude comes from the imaginary part of the one-loop amplitude.

- Let's go back to

$$\tilde{\Gamma}_B^{(n)}(p_1, \dots, p_n; \lambda_B, m_B, \epsilon) = Z_\phi^{-n/2} \tilde{\Gamma}_R^{(n)}(p_1, \dots, p_n; \lambda_R, m_R, \mu, \epsilon).$$

For fixed physics, the LHS is some fixed quantity. The RHS depends on the renormalization point μ and the scheme. The LHS does not! This leads to what is known as the renormalization group equations, which state how the renormalized quantities must vary with μ .

Take $d/d \ln \mu$ of both sides, and use $d\Gamma_B/d\mu = 0$. This gives

$$\left(\frac{\partial}{\partial \ln \mu} + \beta(\lambda_R) \frac{\partial}{\partial \lambda_R} + \gamma_m m_R \frac{\partial}{\partial \ln m_R} - n\gamma \right) \tilde{\Gamma}_R^{(n)}(p_1, \dots, p_n; \lambda_R, m_R, \mu) = 0$$

Here

$$\beta(\lambda) \equiv \frac{d}{d \ln \mu} \lambda_R$$

$$\gamma = \frac{1}{2} \frac{d}{d \ln \mu} \ln Z_\phi$$

$$\gamma_m = \frac{d \ln m_R}{d \ln \mu}.$$

This is the RG equation. Various variants, e.g. Callan-Symanzik equation. It can be integrated, to relate the renormalized Greens functions at different scales μ and μ' . Let us focus on what β and γ mean.

- Understand what β and γ mean: the bare quantities are some function of the renormalized ones and epsilon. E.g. for $\lambda\phi^4$ in MS we have

$$\lambda_B = \mu^\epsilon (\lambda + \delta_\lambda) \equiv \mu^\epsilon \lambda Z_\lambda$$

Let us write

$$Z_\lambda \equiv 1 + \sum_k a_k(\lambda) \epsilon^{-k},$$

where we found $a_1(\lambda) = +3\lambda/16\pi^2$ to one loop. The bare parameter λ_B is independent of μ , whereas λ depends on μ , such that the above relation holds. Take $d/d \ln \mu$ of both sides,

$$0 = \epsilon \lambda Z_\lambda + \beta(\lambda, \epsilon) Z_\lambda + \beta(\lambda, \epsilon) \lambda \frac{dZ_\lambda}{d\lambda}.$$