3/3/10 Lecture 17 outline

• Functional integral for gauge fields. Important point: gauge invariance. Write $A = A_\mu dx^\mu$. Recall gauge symmetry $A \to A^\alpha = A + d\alpha(x)$, with $\psi \to e^{-ie\alpha(x)}\psi$. Redundancy in description, can only observe gauge invariant quantities. Need to replace $\partial_\mu \to D_\mu \equiv \partial_\mu + ieA_\mu$. Then $D_\mu^\alpha \psi^\alpha = e^{-ie\alpha} D_\mu \psi$ transforms nicely, with just an overall phase, and $\bar\psi D_\mu \psi$ is gauge invariant. So the Dirac lagrangian, $\bar\psi(i\not{D} - m)\psi$ is gauge invariant. In functional integral, will have $\int [dA] \exp(iS)$. Integration measure must be gauge invariant, implies it gets a factor of gauge orbit volume. Would like to integrate only over a slice of inequivalent gauge fields, without integrating over the gauge orbits. Need to do this, since otherwise there is no well defined B^{-1}. Recall $S = \int d^4x[-\frac{1}{4}F_{\mu\nu}^2] = \frac{1}{2}\int d^4k A_\mu(x)(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(x)$. Note action vanishes if $\tilde A_\mu(k) = k_\mu \alpha(k)$. Gauge invariance. $A_\mu^T = P_{\mu\nu}A^\nu$, $P_{\mu\nu} = g_{\mu\nu} - \partial_\mu\partial_\nu/\partial^2$. $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}A_\mu^T \partial^2 g^{\mu\nu} A_\nu^T$. Can't invert kinetic terms uniquely to find Green's function. We need to fix the gauge.

The functional integral should be over $\int [dA^\mu]/(GE)$, where we divide by the volume of the gauge equivalent orbits. The gauge equivalent orbits are associated with gauge transformations $\alpha(x)$, e.g. $A_\mu \to A_\mu + \partial_\mu \alpha(x)$ in the Abelian case. We want to do the functional integral over A^μ, dividing out by the $\alpha(x)$.

(Here are some details: Do this via

$$1 = \int [d\alpha(x)]\delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \Delta \int [d\alpha]\delta(G(A^\alpha)),$$

where $G(A) = 0$ is some gauge fixing condition, e.g. Lorentz gauge, $G(A) = \partial_\mu A^\mu$ and

$$\Delta = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)_{G=0}.$$

Δ is the Faddeev-Popov determinant. Write the functional integral as (using the gauge invariance of measure and action)

$$\int [d\alpha][dA]\Delta\delta(G[A])\exp(iS[A]).$$

Have factored out the integral over the group volume. We can then just easily divide out by $[d\alpha]$, just cross it out. What's left is the gauge fixing delta function, and appropriate determinant factor.

Take e.g. $G = \partial^\mu A_\mu - f(x)$ for some function $f(x)$. Then $\Delta \sim \det(\partial^2)$ is a constant. Get

$$e^{iW} = N\int (dA)e^{iS}\delta(\partial^\mu A_\mu - f) = N\int [dA][df]e^{iS}\delta(\partial^\mu A_\mu - f)G(f) = N\int [dA]e^{iS}G(\partial A),$$

for arbitrary functional G. Choose $G(f) = \exp(-\frac{1}{2}i\xi^{-1}\int d^4xf^2)$, for some real number ξ. Get

$$e^{iW} = N\int [dA]\exp(iS - \tfrac{1}{2}\xi^{-1}\int d^4x(\partial^\mu A_\mu)^2).$$

Then get for the propagator

$$D_{\mu\nu} = \frac{-i}{k^2}[g_{\mu\nu} - \frac{k_\mu k_n u}{k^2} + \xi\frac{k_\mu k_\nu}{k^2}].$$

Popular choices: $\xi = 1$ is Feynman propagator, $\xi = 0$ is Landau gauge propagator. Physics is ξ independent (result of gauge invariance, which yields Ward-Takahashi identities). Let's choose to use Feynman gauge.)

- The Ward identity obtained from gauge invariance states that $k_\mu \mathcal{M}^\mu = 0$, where $\mathcal{M}^\mu$ is the part of the amplitude with a external photon line omitted; this ensures that $\epsilon^\mu \to \epsilon^\mu + f(k)k^\mu$ is a symmetry.
- Recall QED Feynman rules, e.g. vertex: $-ie\gamma^\mu$.
- The photon has 1PI propagator $i\Pi^{\mu\nu}(k) = (p^2g^{\mu\nu} - p^\mu p^\nu)\Pi(k^2)$. Summing these gives the full propagator. Writing it in Feynman gauge, get for the full propagator $-ig_{\mu\nu}/p^2(1 - \Pi(p^2))$. Assuming that $\Pi(p^2)$ is regular at $p^2 = 0$, get pole at $p^2 = 0$ with residue $Z_3 \equiv (1 - \Pi(0))^{-1}$.