3/10/10 Lecture 19 and 20 outline

• QED vs QED. In QED, we have gauge invariance $\psi \to e^{ief(x)}\psi$, local $U(1)$ transformations. Generalize to local $SU(N_c)$ gauge transformations: $\psi \to U^f(x)\psi = \exp(igT^a f_a(x))\psi$, where T^a are traceless, Hermitian $N_c \times N_c$ matrices ($a = 1 \dots N_c^2 - 1$), and ψ is a N_c column vector. Gauge conserved color charge. Need covariant derivatives, $\partial_\mu \to D_\mu = \partial_\mu - igA_\mu^a T^a$, i.e. introduce gauge fields, "gluons". The T_a matrices do not commute, $[T^a, T^b] = if_{abc}T^c$: the group is "non-Abelian." (They are normalized b $\mathrm{Tr}T^aT^b = \frac{1}{2}\delta^{ab}$, e.g. for $SU(2)$, $T^a = \sigma^a$, the Pauli matrices.) The effect of this is that the A_μ^a kinetic terms are more complicated. The physics of this is that the gluons carry color charge (unlike the photon, which carries no electric charge).

Gauge transformation: $D_\mu\psi \to D_\mu^f U^f\psi = U^f D_\mu\psi$, i.e. $D_\mu \to UD_\mu U^{-1}$, i.e. $A_\mu^f = UA_\mu^f U^{-1} - ig^{-1}(\partial_\mu U)U^{-1}$.

Field strength: $[D_\mu, D_\nu] = -igF^{\mu\nu}$, i.e. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A^\mu, A^\nu]$, i.e. $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$.

Lagrangian

$$\mathcal{L}_{gaugekinetic} = -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a}, \qquad \mathcal{L}_{ferm} = \bar{\psi}(i\not{D} - m)\psi.$$

Some parts are similar to QED, e.g. the gauge field propagator is $iD_{\mu\nu}^{ab} = \frac{-i\delta^{ab}}{k^2+i\epsilon}(g_{\mu\nu} - (\xi-1)k^\mu k^\nu/k^2)$. Some differences from QED: since gluons are charged, get 3 and 4 gluon diagrams, as seen from expanding $\mathcal{L}_{gaugekinetic}$. These yield added contributions to 1-loop correction to gluon propagator. (We also have to gauge fix and consequently add Faddeev Popov ghosts, e.g. gauge fixing by $G(A) = \partial^\mu A_\mu - \omega(x)$ leads to the FP determinant $\det(\frac{\delta G(A^\alpha)}{\delta\alpha}) \sim \det(\partial^\mu D_\mu)$ and then $\mathcal{L}_{g.f.+ghost} = -\frac{1}{2\xi}(\partial_\mu A^\mu) - c^\dagger\partial^\mu D_\mu c$. Ghosts only appear in closed loops, where the contribution has a minus sign since they're anticommuting fields.)

• Recall $e^+e^- \to \mu^+\mu^-$ at tree level in QED, with total cross section $\sigma = \frac{4\pi\alpha^2}{3s}\sqrt{1 - \frac{m_\mu^2}{s}}(1 + \frac{m_\mu^2}{2s}) \approx \frac{4\pi\alpha^2}{3s}$ at high energy. The total cross section for $e^+e^- \to$ hadrons at high energy is the same, up to a factor of $3\sum_i Q_i^2$, where Q_i accounts for the electric charge of the quarks and 3 accounts for their color. This gave an experimental verification of 3 colors.

• Renormalization.

Consider gauge boson 1PI loop contribution, $i(p^2g^{\mu\nu} - p^\mu p^\nu)\delta^{ab}\Pi(p^2)$. Fermions contribute

$$\Pi(p^2) \supset -\frac{g^2}{16\pi^2}\frac{4}{3}N_f T_2(r)\Gamma(2 - \tfrac{1}{2}d) + \dots.$$

Now add 3 diagrams: two with internal gluons, and one with internal ghost. Each is separately quadratically divergent and would induce a gauge boson mass. But these problems cancel in the sum. The upshot of the sum is

$$\Pi(p^2) \supset -\frac{g^2}{16\pi^2}(-(\frac{13}{6} - \tfrac{1}{2}\xi))C(G)\Gamma(2 - \tfrac{1}{2}d) + \dots .$$

To compute the beta function, must account for loop diagrams involving the fermion vertex. It's somewhat involved (see Peskin). But there is a nice way to determine it from the gauge field propagator in what's known as background field gauge, where one includes a classical background for the field and gauge fixes around that.

Get finally

$$\beta(\alpha) = \frac{\alpha^2}{6\pi}\left(-11N_c + 2N_f\right).$$

(More generally, replace $N_c \to C_2(G)$ and $2N_f \to 4n_f T_2(r)$.) The flavors contribute positively, as in QED. But the colors contribute negatively: they anti-screen charges! So the beta function can be negative, if $11N_c > 2N_f$. This is asymptotic freedom. Integrating the 1-loop result gives

$$\alpha(\mu)^{-1} = \frac{(11N_c - 2N_f)}{6\pi}\ln(\frac{\mu}{\Lambda}).$$

To have $\alpha > 0$, we need $\mu > \Lambda$ (opposite from QED). Note $\alpha(\mu \to \infty) \to 0$, weak in UV = asymptotic freedom. Explains successes of parton model (quarks) for high energy scattering. For QCD, $N_c = 3$, and $N_f = 6$. For energies below the top and bottom mass, use $N_f^{eff} = 4$. Observe e.g. $\alpha(100GeV) \sim 0.1$, so $\Lambda \sim 200MeV$.

On the other hand, $\alpha \to \infty$ for $\mu \to \Lambda$: forces are strong in IR, below scale Λ. Can explain confinement of quarks (there is a million dollar prize, waiting to be collected, if you prove it in detail)!

- Phases of QCD.
- Other topics to mention, anomalies, instantons, etc.