1/22/10 Lecture 6 outline

• Let's write

$$e^{iW[J]} = N\int [d\phi] e^{\frac{i}{\hbar}\left(S[\phi]+\int J\phi\right)},$$

(here we rescaled J by factor of $1/\hbar$ compared with before).

• Suppose diagram has I internal lines, V vertices, L loops. Connected graphs have $L = I - V + 1$. Graphs go like $\hbar^{-V}\hbar^{I} = \hbar^{L-1}$. So $W[J] = W_{-1}\hbar^{-1} + W_0 + \hbar W_1 + \ldots$, where W_{-1} are tree-graphs (no loops), W_0 gives the 1-loop graphs, etc.

• Example: free Klein Gordon theory. We found $Z[J]$ above. Then

$$W[J] = i\tfrac{1}{2}\hbar^{-1}\int d^4x \int d^4y J(X) D_F(x-y) J(y).$$

(Rescaled source J compared with before.)

We see that the only connected Green's function in this case is the 2-point function:

$$G^{(2)}_{free}(x,y) \equiv G(x-y) = \hbar D_F(x-y).$$

In an interacting theory, like $\lambda\phi^4$,

$$G^{(2)}(x,y) = \hbar D_F(x-y) + O(\lambda) \text{ corrections}.$$

• Emphasize that tree graphs are classical. Example: consider $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \phi J$, with the source term J. The classical field EOM is

$$(\partial_\mu\partial^\mu + m^2)\phi_c = -\frac{1}{3!}\lambda\phi_c^3 + J(x).$$

We can solve this classically to zero-th order in λ as

$$\phi_c^{(0)}(x) = \int d^4y D_F(x-y) iJ(y),$$

where $(\partial_\mu\partial^\mu + m^2)D_F(x-y) = -i\delta(x-y)$. To solve to next order in λ, we plug this back into the above:

$$\phi_c^{(1)}(x) = \phi_c^{(0)}(x) - i\frac{1}{3!}\lambda\int d^4y D_F(x-y)\phi_c^{(0)}(y)^3$$

Continue this way, this can be represented as a sum of tree-level diagrams, with one ϕ and different numbers of J's on the external legs. This is perturbation theory for the classical field theory.

• Tree-level contribution $W_{-1}[J]$ and relation to $S[\phi_c]$, via Legendre transform.

• Examples of diagrams contributing to $G^{(n)}_{conn}$ for $n = 2, 4, 6$, in $\lambda\phi^4$.

• We have seen that the loop expansion is an expansion in powers of $\hbar$, since diagrams go like $\hbar^{L-1}$. Question: are we expanding in $\hbar$ (loops), or in powers of the small coupling constants, or both? Answer: it's generally the same expansion. Consider e.g. $\lambda\phi^r$ interaction. Then a connected diagram with E external lines (amputating their propagators) and I internal lines and V vertices is $\sim \hbar^{I-V}\lambda^V$. Now we use $L = I - V + 1$ and $E + 2I = rV$ (conservation of ends of the lines) to get that the diagram is $\sim \left(\hbar\lambda^{2/(r-2)}\right)^{L-1}\lambda^{E/(r-2)}$, so for fixed E the loop expansion is an expansion in powers of the effective coupling $\alpha \sim \hbar\lambda^{2/r-2}$.

• It is useful to define a further specialization of the diagrams, those that are 1PI: one - particle irreducible. The definition is that the diagrams is connected, and moreover remains connected upon removing any one internal progagator (and amputating all external legs).

•Examples of $n = 2, 4, 6$ point 1PI diagrams in $\lambda\phi^4$.

• In momentum space, it is defined from the 1PI diagram, with all external momenta taken to be incoming:

$$\text{1PI diagram} \equiv i\tilde{\Gamma}^{(n)}(p_1, \ldots p_n),$$

where the external propagators are amputated, and the $(2\pi)^4\delta^4(\sum_i p_i)$ is omitted. If there is an interaction like $g\phi^n/n!$, then, at tree-level, $\tilde{\Gamma}^{(n)} = g$. Special definition for case $n = 2:$ we define the 1PI diagram to be $-i\Pi'(p)$, and we instead define

$$i\tilde{\Gamma}^{(2)}(p, -p) = \text{1PI diagram} + i(p^2 - m^2) = i(p^2 - m^2 - \Pi'(p^2)).$$

Define position space 1PI diagrams by Fourier transform. They correspond to

$$\Gamma^{(n)}(x_1, \ldots x_n) = \langle T\phi(x_1)\ldots\phi(x_n)\rangle|_{1PI}.$$

• 2-point function, via summing geometric series:

$$D(p) = \frac{i}{\tilde{\Gamma}^{(2)}} = \frac{i}{p^2 - m^2 - \Pi'(p^2)}.$$

$-i\Pi'$ is computed from the 1PI diagrams. $\Pi'(p^2)$ is called the self-energy, like momentum dependent mass term. The special definition of $\tilde{\Gamma}^{(2)}$ is because $D(p) = i/\tilde{\Gamma}^{(2)}$ will be nice, and allow extending to higher point functions.

• The point of the 1PI diagrams is that the quantum loop corrections are simply obtained by replacing the vertices with the 1PI greens functions! Indeed, Draw pictures for $n = 2, 4, 6$ point functions. Obtain the full $W[J]$ via tree-graphs assembled from the 1PI building blocks.

• Note that there are are no tree level IPI diagrams for $\tilde{\Gamma}^{(n)}$ except for $n = 4$ in $\lambda\phi^4$, so $\tilde{\Gamma}^{(n)} = -\lambda\hbar^{-1}\delta_{n,4} + \mathcal{O}(\hbar^0) + \ldots$. At order $\hbar^0$, i.e. 1-loop, note that there are terms for all even n. There can not be terms for odd n, because of the $\phi \to -\phi$ symmetry.

• There is also a generating function for the 1PI green's functions:

$$\Gamma[\phi] = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \Gamma^{(n)}(x_1, \ldots x_n)\phi(x_1) \ldots \phi(x_n).$$

This quantity is called the effective action. Find that

$$\Gamma[\phi] = \frac{1}{\hbar} \left(S[\phi] + \mathcal{O}(\hbar) \right).$$

E.g. in $\lambda\phi^4$, $\Gamma[\phi] = \hbar^{-1} \int d^4x [\frac{1}{2}\phi(-\partial^2 - m^2)\phi - \frac{1}{4!}\lambda\phi^4]$+(quantum corrections). The quantum corrections are e.g. corrections to the mass from $m^2 \to m^2 + \hbar\Pi'(p^2)$, a correction to λ at order $\hbar$, and higher powers of ϕ at order $\hbar^{-1}(\hbar^L)$ for $L \geq 1$.

• Connecting $\Gamma[\phi]$ and $W[J]$. Introduce a (to count loops, formally take $a \to 0$):

$$e^{iW[J,a]} \equiv N \int [d\phi] e^{i(\Gamma[\phi] + \int d^4x J\phi)/a}.$$

Then LHS=$\exp(i(W[J] + O(a))/a)$. Evaluate RHS by stationary phase:

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -J(x) \qquad \text{for} \quad \phi = \overline{\phi}(x),$$

which is some functional of J. So the RHS is

$$N e^{i(\Gamma[\overline{\phi}] + \int d^4x J\overline{\phi} + \mathcal{O}(\sqrt{a}))}.$$

Conclude

$$W[J] = \Gamma[\overline{\phi}] + \int d^4x J(x)\overline{\phi}(x).$$

This is a Legendre transform. Like $F = E - TS$ in Stat Mech. There is also the inverse transform:

$$\Gamma[\overline{\phi}] = W[J] - \int d^4x J(x)\overline{\phi}(x).$$

$\overline{\phi}(x)$ can be interpreted as the average of $\phi(x)$ in the presence of the source; sometimes called classical field:

$$\overline{\phi}(x) = \frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}.$$

The functional derivatives of $\Gamma[\overline{\phi}]$, upon setting $\overline{\phi} = 0$, give $\Gamma^{(n)}(x_1, \ldots x_n)$. In particular,

$$\frac{\delta\Gamma[\phi_c]}{\delta\overline{\phi}(x)}|_{\overline{\phi}=0} = \Gamma^{(1)}(x) = 0.$$