1/27/10 Lecture 7 outline

- Last time: 1PI diagrams $\widetilde{\Gamma}^{(n)}(p_1, \dots p_n)$. Recall 2-point function is a special case:

$$D(p) = \frac{i}{\tilde{\Gamma}^{(2)}} = \frac{i}{p^2 - m^2 - \Pi'(p^2)}$$

where $-i\Pi'$ is computed from the 1PI diagrams.

Mentioned generating function for the 1PI green's functions:

$$\Gamma[\phi] = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \Gamma^{(n)}(x_1, \dots x_n)\phi(x_1)\dots\phi(x_n).$$

This quantity is called the effective action. Find that

$$\Gamma[\phi] = \frac{1}{\hbar}\left(S[\phi] + \mathcal{O}(\hbar)\right).$$

E.g. in $\lambda\phi^4$, $\Gamma[\phi] = \hbar^{-1}\int d^4x[\frac{1}{2}\phi(-\partial^2 - m^2)\phi - \frac{1}{4!}\lambda\phi^4]$+(quantum corrections). The quantum corrections are e.g. corrections to the mass from $m^2 \to m^2 + \hbar\Pi'(p^2)$, a correction to λ at order $\hbar$, and higher powers of ϕ at order $\hbar^{-1}(\hbar^L)$ for $L \geq 1$.

Key fact: all connected diagrams are contained in the *tree-level* diagrams computed using the propagators and vertices coming from the effective action $\Gamma[\phi]$.

To make this precise, we can connect $\Gamma[\phi]$ and $W[J]$. Since both have $\hbar$ inside, introduce a new parameter a (to count loops, formally take $a \to 0$):

$$e^{iW[J,a]} \equiv N \int [d\phi] e^{i(\Gamma[\phi] + \int d^4x J\phi)/a}.$$

Then LHS=exp$(i(W[J] + O(a))/a)$. Evaluate RHS by stationary phase:

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -J(x) \qquad \text{for} \quad \phi = \overline{\phi}(x),$$

which is some functional of J. So the RHS is

$$N e^{i(\Gamma[\overline{\phi}] + \int d^4x J\overline{\phi} + \mathcal{O}(\sqrt{a}))}.$$

Conclude

$$W[J] = \Gamma[\overline{\phi}] + \int d^4x J(x)\overline{\phi}(x).$$

This is a Legendre transform. Like $F = E - TS$ in Stat Mech. There is also the inverse transform:

$$\Gamma[\overline{\phi}] = W[J] - \int d^4x J(x)\overline{\phi}(x).$$

$\overline{\phi}(x)$ can be interpreted as the average of $\phi(x)$ in the presence of the source; sometimes called classical field:

$$\overline{\phi}(x) = \frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}.$$

The functional derivatives of $\Gamma[\overline{\phi}]$, upon setting $\overline{\phi} = 0$, give $\Gamma^{(n)}(x_1, \ldots x_n)$. In particular,

$$\frac{\delta\Gamma[\phi_c]}{\delta\overline{\phi}(x)}\Big|_{\overline{\phi}=0} = \Gamma^{(1)}(x) = 0.$$

Recall from last time that we have $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \phi J$, with the source term J. The classical field EOM is

$$(\partial_\mu\partial^\mu + m^2)\phi_c = -\frac{1}{3!}\lambda\phi_c^3 + J(x).$$

As discussed last time, we can solve this in perturbation theory in λ, with only tree-level diagrams. The generating functional for tree-level diagrams is $W_c[J] = S[\phi_c] + \int d^4x J\phi_c$.

The field $\overline{\phi}$ satisfies the same equation, up to order $\hbar$ corrections:

$$(\partial_\mu\partial^\mu + m^2)\overline{\phi} = -\frac{1}{3!}\lambda\overline{\phi}^3 + J(x) + \mathcal{O}(\hbar).$$

So, at the classical level, $\phi_c = \overline{\phi}$. But $\overline{\phi}$ includes the quantum loop corrections.

• One-loop effective potential for $\lambda\phi^4$. The effective potential is found from $\Gamma[\phi]$, keeping the terms with no derivatives. Find

$$\begin{aligned} V_1(\phi) &= i\sum_{n=1}^{\infty}\frac{1}{2n}\int\frac{d^4k}{(2\pi)^4}\left(\lambda\frac{1}{k^2-m^2+i\epsilon}\frac{\phi^2}{2}\right)^n \\ &= \tfrac{1}{2}\int\frac{d^4k_E}{(2\pi)^4}\ln\left(1+\frac{\frac{1}{2}\lambda\phi^2}{k_E^2+m^2}\right) \end{aligned}$$

(S. Coleman and E. Weinberg.) Symmetry factors: $1/n!$ not all the way cancelled, because of Z_n rotation symmetry, and reflection, gives $1/2n$. At each vertex, can exchange external lines, so $1/4!$ not all the way cancelled, leads to $1/2$ for each vertex. Still have to explain how to handle k_E integral. We'll come back to this later.

• Let's consider the 1-loop term in $\tilde{\Gamma}^{(2)}$ for $\lambda\phi^4$. Get

$$-i\Pi'(p^2) = (-i\lambda)\tfrac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2} + \text{more loops}.$$

Now rotate to Euclidean space, $d^4k = id^4k_E$,

$$\Pi'(p^2) = \tfrac{1}{2}\lambda \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} + \text{more loops}.$$

Recall expression $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of a unit sphere S^{D-1}. For $D = 4$, get $\Omega_3 = 2\pi^2$, so

$$\Pi'(p^2) = \frac{\lambda m^2}{32\pi^2} \int_0^{\Lambda^2/m^2} \frac{udu}{u+1} = \frac{\lambda m^2}{32\pi^2} \left(\frac{\Lambda^2}{m^2} - \log(1 + \frac{\Lambda^2}{m^2}) \right).$$

Here Λ is a UV momentum cutoff. Result is quadratically (and also log) divergent as $\Lambda \to \infty$. The subject of renormalization is the physical interpretation of these divergences. The first thing to do is to regulate them, as we did above with a momentum cutoff. There are other ways to regulate. How one regulates is physically irrelevant. The physics is in the renormalization interpretation of the regulated results, and at the end of the day the choice of regulator doesn't matter.