1/3/11 Lecture outline

⋆ Last quarter's class covered *Peskin and Schroeder* chapters 1-5. The next topic in *Peskin and Schroeder* is radiative corrections (Feynman diagrams with loops), particularly in QED. We'll get to this, but after a detour on the Feynman path integral. Also, before discussing QED, we'll discuss renormalization, illustrating it in scalar field theory. We'll discuss topics contained in *Peskin and Schroeder* chapters 6-13, but in a different order than in the text.

Week 1 reading: *Peskin and Schroeder* sections 9.1, 9.2, 9.3.

• We'll focus for a while on scalar field theory, e.g. $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$, with e.g. $V(\phi) = \frac{1}{2}m^2\phi^2 + V_{int}(\phi)$, with e.g. $V_{int}(\phi) = \frac{1}{4!}\lambda\phi^4$.

• Fields can be quantized using canonical quantization, as was discussed last quarter. The following is a summary for those who want a brief review. The field $\phi(x)$ is analogous to $q(t)$ in QM (indeed, QM is a particular case of QFT in one dimension), and its conjugate momentum is $\Pi = \partial\mathcal{L}/\partial\dot{\phi}$. These are operators, with equal time commutators

$$[\phi(t,\vec{x}), \Pi(t,\vec{x}')] = i\hbar\delta^3(\vec{x}-\vec{x}').$$

We'll usually set $\hbar = 1$. The S-matrix elements, used to compute scattering cross sections and lifetimes etc. are computed from an amplitude $\langle f|S|i\rangle$ which is related to the vacuum expectation values of time-ordered products of the fields. This is seen from Dyson's formula or from the LSZ derivation, to be discussed in more detail later:

$$\begin{aligned}\langle f|i\rangle &= \langle k_{1'}\dots k_{n'}|k_1\dots k_n\rangle \\ &= i^{n+n'}\prod_{j'=1}^{n'}\int d^4x'_j e^{ik'_jx'_j}(\partial^2_{j'}+m^2)\prod_{j=1}^{n}e^{-ik_jx_j}(\partial^2_j+m^2)G_{n+n'}(x_1\dots x_n, x_{1'}\dots x_{n'}),\end{aligned} \tag{1}$$

where

$$G_{n+n'}(x_1\dots x_n, x_{1'}\dots x_{n'}) \equiv \langle 0|T\phi(x_{1'})\dots\phi(x_{n'})\phi(x_1)\dots\phi(x_n)|0\rangle. \tag{2}$$

Using Wick's theorem,

$$T(\phi_1\dots\phi_n) =: \phi_1\dots\phi_n : + : \text{all contractions},$$

then led to a derivation of Feynman's rules for computing amplitudes, from Feynman diagrams.

• Our first topic will be to get an alternative derivation of the Feynman rules, using the Feynman path integral. This gives an alternative to canonical quantization for quantizing particles and fields, and additional insights into the Feynman diagrams and rules.

We'll start with considering particle quantum mechanics. The probability amplitude to go from position q at time t to q' at time t' is $\langle q', t'|q, t\rangle$. Let's write this as the time evolution operator

$$U(x_a, x_b; T) = \langle x_b|e^{-iHT/\hbar}|x_a\rangle.$$

Satisfies SE

$$i\hbar\partial_T U = HU.$$

Feynman:

$$U(x_a, x_b; T) = \int [dx(t)]e^{iS[x(t)]/\hbar}.$$

Integral can be broken into time slices, as way to define it.

E.g. free particle

$$\left(\frac{-im}{2\pi\hbar\epsilon}\right)^{N/2}\int\prod_{i=1}^{N-1}dx_i \exp[\frac{im}{2\hbar\epsilon}\sum_{i=1}^{N}(x_i - x_{i-1})^2]$$

Where we take $\epsilon \to 0$ and $N \to \infty$, with $N\epsilon = T$ held fixed.

Do integral in steps. Apply expression for real gaussian integral (valid: analytic continuation):

$$\int_{-\infty}^{\infty} d\phi \exp(ia\phi^2) = \sqrt{\frac{i\pi}{a}}.$$

where we analytically continued from the case of an ordinary gaussian integral. Think of a as being complex. Then the integral converges for $\mathrm{Im}(a) > 0$, since then it's damped. To justify the above, for real a, we need the integral to be slightly damped, not just purely oscillating. To get this, take $a \to a + i\epsilon$, with $\epsilon > 0$, and then take $\epsilon \to 0^+$. We'll see that this is related to the $i\epsilon$ that we saw last quarter in the Feynman propagator, which gave the T ordering.

After $n - 1$ steps, get integral:

$$\left(\frac{2\pi i\hbar n\epsilon}{m}\right)^{-1/2} \exp[\frac{m}{2\pi i\hbar n\epsilon}(x_n - x_0)^2].$$

So the final answer is

$$U(x_b, x_a; T) = \left(\frac{2\pi i\hbar T}{m}\right)^{-1/2} \exp[im(x_b - x_a)^2/2\hbar T].$$

Note that the exponent is $e^{iS_{cl}/\hbar}$, where S_{cl} is the classical action for the classical path with these boundary conditions. (More generally, get a similar factor of $e^{iS_{cl}/\hbar}$ for interacting theories, from evaluating path integral using stationary phase.)

Plot phase of Uas a function of $x = x_b - x_a$, fixed T, Lots of oscillates. For large x, nearly constant wavelength λ, with

$$2\pi = \frac{m(x+\lambda)^2}{2\hbar T} - \frac{2m^2}{2\hbar T} \approx \frac{mx\lambda}{\hbar T} = p\lambda/\hbar.$$

Gives $p = \hbar k$.

Recover $\psi \sim e^{ipx/\hbar}$. More generally, get $p = \hbar^{-1}k$, with $p = \partial S_{cl}/\partial x_b$ (can show $p = \partial L/\partial \dot{x} = \partial S_{cl}/\partial x_b$. Can also recover $\psi \sim e^{-i\omega T}$, with $\omega = \hbar^{-1}(-\partial S_{cl}/\partial t_b)$. Agrees with $E = \hbar\omega$, since $E = p\dot{x} - L = -\partial S_{cl}/\partial t_b$.

• Nice application: Aharonov-Bohm. Recall $L = \frac{1}{2}m\dot{\vec{x}}^2 + q\dot{\vec{x}} \cdot \vec{A} - q\phi$. Solenoid with $B \neq 0$ inside, and $B = 0$ outside. Phase difference in wavefunctions is

$$e^{i\Delta S/\hbar} = e^{iq\oint \vec{A}\cdot d\vec{x}/\hbar} = e^{iq\Phi/\hbar}.$$

Aside on Dirac quantization for magnetic monopoles.

• Can derive the path integral from standard QM formulae, with operators, by introducing the time slices and a complete set of q and p eigenstates at each step.

$$\langle q', t'|q, t\rangle = \int\int \prod_{j=1}^{N} dq_j \langle q'|e^{-iH\delta t}|q_{N-1}\rangle\langle q_{N-1}|e^{-iH\delta t}|q_{N-2}\rangle \ldots \langle q_1|e^{-iH\delta t}|q\rangle,$$

where we'll take $N \to \infty$ and $\delta t \to 0$, holding $t' - t \equiv N\delta t$ fixed. Note that even though $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]+\ldots}$, we're not going to have to worry about this for $\delta t \to 0$: $e^{-iH\delta t} = e^{-i\delta t p^2/2m} e^{-i\delta t V(q)} e^{\mathcal{O}(\delta t^2)}$. Now note

$$\langle q_2|e^{-iH\delta t}|q_1\rangle = \int dp_1 \langle q_2|e^{-iH\delta t p^2/2m}|p_1\rangle\langle p_1|e^{-iV(q)\delta t}|q_1\rangle,$$

$$= \int dp_1 e^{-iH(p_1,q_1)\delta t} e^{ip_1(q_2-q_1)}.$$

This leads to

$$\langle q', t'|q, t\rangle = \int [dq(t)][dp(t)] \exp(i\int_t^{t'} dt(p(t)\dot{q}(t) - H(p,q))),$$

and taking H quadratic in momentum and doing the p gaussian integral recovers the Feynman path integral.

• The same derivation as above leads to e.g.

$$\langle q_4, t_4 | T\widehat{q}(t_3)\widehat{q}(t_2)|q_1, t_1\rangle = \int [dq(t)] q(t_3) q(t_2) e^{iS/\hbar},$$

where the integral is over all paths, with endpoints at (q_1, t_1) and (q_4, t_4).

A key point: the functional integral automatically accounts for time ordering! Note that the LHS above involves time ordered operators, while the RHS has a functional integral, which does not involve operators (so there is no time ordering). The fact that the time ordering comes out on the LHS is wonderful, since know that we'll need to have the time ordering for using Dyson's formula, or the LSZ formula, to compute quantum field theory amplitudes.

• The nice thing about the path integral is that it generalizes immediately to quantum fields, and for that matter to all types (scalars, fermions, gauge fields). E.g.

$$\langle \phi_b(\vec{x}, T)|e^{-iHT}|\phi_a(\vec{x}, 0)\rangle = \int [d\phi] e^{iS/\hbar} \qquad S = \int d^4x \mathcal{L}.$$

This is then used to compute Green's functions:

$$\langle \Omega | T \prod_{i=1}^{n} \phi_H(x_i) | \Omega \rangle = Z_0^{-1} \int [d\phi] \prod_{i=1}^{n} \phi(x_i) \exp(iS/\hbar),$$

with $Z_0 = \int [d\phi] \exp(iS/\hbar)$. Again, as noted above, the T ordering will be automatic.