1/19/11 Lecture 5 outline

• Last time, $\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \hbar J\phi$, and r

$$Z_{free}[J] = Z_0[J] = \exp(-\tfrac{1}{2}\hbar \int d^4x d^4y J(x) D_F(x-y) J(y)), \tag{1}$$

with $\phi(x) \to \frac{1}{i}\frac{\delta}{\delta J(x)}$. Including interactions,

$$Z[J] = N \exp[\frac{i}{\hbar} S_{int}[-i\frac{\delta}{\delta J}]) Z_{free}[J], \tag{2}$$

where N is an irrelevant normalization factor (independent of J).

$$\begin{aligned} G^{(n)}(x_1 \dots x_n) &= \frac{\int [d\phi]\phi(x_1)\dots\phi(x_n)\exp(\frac{i}{\hbar}S_I[\phi])\exp[\frac{i}{\hbar}S_{free}]}{\int [d\phi]\exp(\frac{i}{\hbar}S_I[\phi])\exp[\frac{i}{\hbar}S_{free}]}, \\ &= \frac{1}{Z[J]}\prod_{j=1}^{n}\left(-i\hbar\frac{\delta}{\delta J(x_j)}\right)\cdot Z[J]\big|_{J=0}. \end{aligned}$$

(The denominator (in both lines) cancels off the vacuum bubble diagrams, which don't depend specifically on the Green's function.)

• Illustrate the above formulae, and relation to Feynman diagrams, e.g. $G^{(1)}$, $G^{(2)}$ and $G^{(4)}$ in $\lambda\phi^4$ theory. The functional integral accounts for all the Feynman diagrammer; even symmetry factors etc. come out simply from the derivatives w.r.t. the sources, and the expanding the exponentials,

$$G^{(n)}(x_1, \dots x_n) = \frac{1}{Z[J]}\prod_{j=1}^{n}(-i\frac{\delta}{\delta J(x_j)})\sum_{N=1}^{\infty}\frac{1}{N!}\left(-i\frac{\lambda}{4!\hbar}\int d^4y(-i)^4\frac{\delta^4}{\delta J(y)^4}\right)^N Z_0[J]\big|_{J=0}.$$

etc. Consider, for example, the 4-point function $G^{(4)}(x_1, x_2, x_3, x_4) \equiv \langle T\phi(x_1)\dots\phi(x_4)\rangle/\langle 0|0\rangle$ in $\frac{\lambda_4}{4!}\phi^4$. So take 4-fuctional derivatives w.r.t. the source, at points $x_1 \dots x_4$, i.e. $\delta/\delta J(x_1)\dots\delta/\delta J(x_4)$. The $\mathcal{O}(\lambda^0)$ term thus comes from expanding the exponent to quadratic order (4 J's), corresponding to the disconnected diagrams with two propagators. Each propagator ends on a point x_i. This is like the 4-point function in the SHO homework. Now consider the $\mathcal{O}(\lambda)$ contribution, coming from expanding out the interaction part of the exponent in (2) to $\mathcal{O}(\lambda)$. There are now 4 extra $\delta/\delta J(y)$, for a total of 8, so the contributing term comes from expanding the exponent in (1) to 4-th order, i.e. there are 4 propagators. This gives the connected term, along with several disconnected terms. Go through these terms and their combinatorics.

• We got the functional integral to converge via the $i\epsilon$, recall it came from evaluating the gaussian functional integrals like $\int_{-\infty}^{\infty} d\phi e^{ia\phi^2}$, taking a =real $+i\epsilon$. There is another way, which is often very useful: Wick rotate to Euclidean space. The k_0 momentum integral, like that in

$$D_F(x) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon}$$

can be analytically continued, as long as no poles are crossed. We can "Wick rotate" the dk_0 by $+\pi/2$, so k_0 runs from $-i\infty$ to $+i\infty$ along the imaginary axis. This allows continuation to $k_0 = ik_4$, with k_4 real, and k_4 runs from $-\infty$ to $+\infty$. So $k^2 = -k_E^2$, and $d^4k = id^4k_E$. To avoid having e^{ikx} blow up anywhere, we also continue time: $x_0 = -ix_4$, so $d^4x = -id^4x_E$. The Feynman propagator, in Euclidean space, is

$$\Delta_E(x) = \int \frac{d^4k_E}{(2\pi)^4} e^{-ikx} \frac{1}{k_E^2 + m^2},$$

where we can now drop the $i\epsilon$, since it's no longer needed. Note $k_E^2 + m^2$ is never zero, so the integrand never has a pole, and the solution Δ_E is unique.

The action changes as $S = \int d^4x(\frac{1}{2}\partial\phi\partial\phi - V) = i\int d^4x_E(\frac{1}{2}\partial_{x_E}\phi\partial_{x_E} + V) = iS_E$, where S_E looks like the energy now, $S_E =$ "H"! Then

$$\int [d\phi] \exp[\frac{i}{\hbar}S] \to \int [d\phi] e^{-\frac{1}{\hbar}\text{“}H\text{”}}$$

which is like the partition function of stat mech (as you saw in your HW)! (But here "H" is like the Hamiltonian of a theory living in 4 spatial dimensions..). Note $\hbar$ here appears as does T (temperaure) there, connects intuition of quantum fluctuations with intuition of thermal fluctuations!

It is sometimes useful to do all Feynman diagram computations in Euclidean space, and analytically continue back to Minkowski at the end of the day. So

	Mink	Euc
propagator	$\frac{i}{k^2-m^2} = \frac{-i}{k_E^2+m^2}$	$\frac{1}{k_E^2+m^2}$
vertex	$-ig$	$-g$
loop	$\int \frac{d^4k}{(2\pi)^4} = i\int \frac{d^4k_E}{(2\pi)^4}$	$\int \frac{d^4k_E}{(2\pi)^4}$.

Comparing with what we had before, we have dropped some factors of i:

$$i^{L+V-I} = i,$$

since (connected) diagrams have $L = I - V + 1$. So every diagram in the sum just differs by a factor of i, so the sums work the same as before (no relative differences).

• We normalize $Z[J = 0] = 1$, since we anyway divide by the vacuum-to-vacuum amplitude. This recovers the story of cancellation of bubble diagrams. For computing S-matrix elements, we will especially be interested in *connected* Green's functions. There are nice combinatoric formulae (you might have already seen some last quarter?). E.g.

$$\sum \text{all diagrams} = \left(\sum \text{“connected”}\right) \cdot \exp\left(\sum \text{disconnected vacuum bubbles}\right).$$

And the vacuum bubble diagrams cancel. We write “connected” because for $n > 2$ point functions there are still disconnected diagrams, connected to the external points, included in this sum. But even those disconnected diagrams drop out when we consider $S - 1$: they correspond to the 1. In the end, we're interested in the fully connected diagrams. There is a generating functional for them. (N.B. sometimes people reverse the names of what I'm calling W and Z!. Peskin calls $W \to E$.) Defining,

$$iW[J] \equiv \ln Z[J]$$

$iW[J]$ is the generating functional for connected Green's functions

$$G^{(n)}_{conn}(x_1, \ldots x_n) = \hbar^{-1} \prod_{j=1}^{n} \frac{-i\delta}{\delta J(x_j)} iW[J],$$

i.e.

$$iW[J] = \hbar \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n G^{(n)}_{conn}(x_1, \ldots x_n) J(x_1) \ldots J(x_n).$$

In momentum space, we can write:

$$iW[J] = \hbar \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \ldots \frac{d^4k_n}{(2\pi)^4} \tilde{J}(-k_1) \ldots \tilde{J}(-k_n) \tilde{G}_c(k_1, \ldots k_n).$$

• Examples, to illustrate how $iW[J] \equiv \ln Z[J]$ gives the connected diagrams. First consider the 1-point function

$$-i\frac{\delta iW}{\delta J} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J} \equiv \phi_{cl}(x).$$

Picture this diagrammatically as a propagator connecting the point x to a blob, where the blob represents a $\sum_n \lambda^n$ sum of diagrams. Note that there are no disconnected diagrams, thanks to the denominator above which subtracts out the disconnected vacuum bubble diagrams.