1/24/11 Lecture 6 outline

- Examples of diagrams contributing to $G^{(n)}_{conn}$ for $n = 2, 4, 6$, in $\lambda\phi^4$.
- Last time, define a generating functional $iW[J] \equiv \ln Z[J]$, i.e.

$$e^{iW[J]} = N \int [d\phi] e^{\frac{i}{\hbar}\left(S[\phi]+\hbar\int J\phi\right)}.$$

We went back to defining the source J such that $\phi(x) \to -i\hbar\frac{\delta}{\delta J(x)}$. As we'll now motivate, it turns out that W is the generating functional for the *connected* Greens functions:

$$iW[J] = \hbar \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n G^{(n)}_{conn}(x_1, \ldots x_n)\hbar^{-n} J(x_1)\ldots J(x_n).$$

In momentum space, we can write:

$$iW[J] = \hbar \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \tilde{J}(-k_1)\ldots\tilde{J}(-k_n)\hbar^{-n}\tilde{G}_c(k_1, \ldots k_n).$$

Will later recall LSZ: how to relate Green's functions to S-matrix elements (and hence physical observables). As seen there, only connected diagrams contribute; this is why W is useful.

Examples, to illustrate how $iW[J] \equiv \ln Z[J]$ gives the connected diagrams. First consider the 1-point function

$$-i\frac{\delta iW}{\delta J} = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J} \equiv \phi_{cl}(x).$$

Picture this diagrammatically as a propagator connecting the point x to a blob, where the blob represents a $\sum_n \lambda^n$ sum of diagrams. Note that there are no disconnected diagrams, thanks to the denominator above which subtracts out the disconnected vacuum bubble diagrams.

Now consider the two point function

$$(-i)^2 \frac{\delta^2}{\delta J(x)\delta J(y)}(iW) = \langle \phi(x)\phi(y)\rangle_J - \langle\phi(x)\rangle_J\langle\phi(y)\rangle_J.$$

Note that $\langle\phi(x)\phi(y)\rangle$ has two types of contributions, connected and disconnected; the 2nd term precisely cancels off the disconnected ones. The connected one is pictured as a line connecting x and y, with a single blob propagator, whereas the disconnected contribution has two disconnected blobs. Similarly $\delta W/\delta J^3$ has terms like $\langle\phi\phi\phi\rangle - (\langle\phi\phi\rangle\langle\phi\rangle + 2\ terms) +$

$2\langle\phi\rangle\langle\phi\rangle\langle\phi\rangle$, which give precisely $\langle\phi\phi\phi\rangle_{connected}$. Can prove by induction that the *log* in W properly subtracts away all non-connected diagrams!

• Let's consider the powers of $\hbar$. Example: free Klein Gordon theory. We found

$$W[J] = i\tfrac{1}{2}\hbar^{-1}\int d^4x \int d^4y J(X) D_F(x-y)J(y).$$

We see that the only connected Green's function in this case is the 2-point function:

$$G^{(2)}_{free}(x,y) \equiv G(x-y) = \hbar D_F(x-y).$$

So the propagator contains a factor of $\hbar$. In an interacting theory, like $\lambda\phi^4$,

$$G^{(2)}(x,y) = \hbar D_F(x-y) + O(\lambda) \text{ corrections.}$$

• In an interacting theory, the vertices have factors like $-i\lambda/\hbar$, while the proagators are proportional to $\hbar$. Suppose a diagram has I internal lines, V vertices, L loops. Connected graphs have $L = I - V + 1$. Graphs go like $\hbar^{-V}\hbar^{I} = \hbar^{L-1}$. So $W[J] = W_{-1}\hbar^{-1} + W_0 + \hbar W_1 + \ldots$, where W_{-1} are tree-graphs (no loops), W_0 gives the 1-loop graphs, etc.

• Consider $W_{-1}[J]$, the leading term in the $\hbar \to 0$ limit. In this limit, the functional integral localizes on the classical path, so

$$W_{-1}[J] = S[\phi_c] + \int \phi_c J.$$

• Emphasize that tree graphs are classical. Example: consider $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \phi J$, with the source term J. The classical field EOM is

$$(\partial_\mu\partial^\mu + m^2)\phi_c = -\frac{1}{3!}\lambda\phi_c^3 + J(x).$$

We can solve this classically to zero-th order in λ as

$$\phi_c^{(0)}(x) = \int d^4y D_F(x-y) iJ(y),$$

where $(\partial_\mu\partial^\mu + m^2)D_F(x-y) = -i\delta(x-y)$. To solve to next order in λ, we plug this back into the above:

$$\phi_c^{(1)}(x) = \phi_c^{(0)}(x) - i\frac{1}{3!}\lambda\int d^4y D_F(x-y)\phi_c^{(0)}(y)^3$$

Continue this way, this can be represented as a sum of tree-level diagrams, with one ϕ and different numbers of J's on the external legs. This is perturbation theory for the classical field theory.

• To summarize the above, we solve $\frac{\delta}{\delta\phi}(S[\phi] + \int J\phi)|_{\phi=\phi_c} = 0$ for $\phi_c[J]$. Here we plugged the solution $\phi_c[J]$ back in to the action and source term, to get $W_{-1}[J] = S[\phi_c] + \int \phi_c J$. The LHS depends on J but not ϕ_c; indeed, we solve for ϕ_c by $\frac{\delta}{\delta\phi_c} W_{-1}[J] = 0$. Conversely $S[\phi_c]$ does not depend on J. Indeed,

$$\phi_c = \frac{\delta}{\delta J} W_{-1}[J], \qquad J = -\frac{\delta}{\delta\phi_c} S[\phi_c]$$

which fits with $\frac{\delta}{\delta J} S[\phi_c] = 0$. $\phi_c = \frac{\delta}{\delta J} W_{-1}[J]$ is the classical limit of $\phi_{cl}(x) \equiv \langle 0|\phi|0\rangle_J / \langle 0|0\rangle_J$.

This is a Legendre transform, between $\phi_c(x)$ and $J(x)$. Recall e.g. in thermodynamics, $dE = TdS - PdV$, so $E = E(S, V)$, and then can define e.g. $E + PV = H(S, P)$, so adding PV to E changes it from being a function of V to being a function of P, with $P = -\partial E/\partial V$ and $V = \partial H/\partial P$. Likewise, above, for $S[\phi_c]$ vs $W_{-1}[J]$.