1/11/12 Lecture outline

$\star$ See lecture notes for details. Continue where we left off last time, with relativity. End of last time: Events are lightlike, spacelike, timelike separated for $ds^2 = 0, > 0, < 0$.

• The Δs^2 between two events is an example of a 4-scalar, a quantity that's the same for all inertial observers. As we said, the principle of relativity posits that the result of any experiment is the same in any inertial frame of reference. Every physical quantity is either a 4-scalar or an appropriate generalization: 4-vector, 4-tensor, that can be used to form frame-invariant physical quantities.

• The action should be a 4-scalar. Then the equations of motion, coming from least action, are guaranteed to be properly related in different frames of reference.

• Another 4-scalar: the total electric charge of an object, or contained within some "box" in space.

• Another 4-scalar: the proper time interval between two time-like separated events A and B: $\Delta\tau = \int_A^B d\tau$, where $d\tau^2 \equiv ds^2/c^2$.

• Recall condition on L for Gallean invariance under $\vec{x}_i = \vec{x}_i' + \vec{v}_0 t$, $\vec{v}_i = \vec{v}_i' + \vec{v}_0$:

$$L(\vec{x}_i, \vec{v}_i, t) = L(\vec{x}_i', \vec{v}_i', t') + \frac{dG}{dt},$$

which implies that L is linear in the $\langle v\rangle_i^2$, $\partial L/\partial \vec{v}_i^2 = \alpha_i$ a constant. Consider $\vec{v}_0$ small, show how it works with $G = \sum_i 2\vec{x}_i \cdot \vec{v}_0/\alpha_i$. Introduce mass: $\alpha_i = m_i/2$. Writing the EL equations, get forces from gradients of U: infinite signal speed. Not correct: maximum actual signal speed is c (or the speed of superluminal neutrinos ?!? .... just kidding).

• More on proper time: read by moving clock. For timelike separated events, there is a frame where the events occur at the same spatial position. The proper time is the time experienced by clocks in that frame. So $d\tau = dt'$ when $d\vec{x}' = 0$.

• Four vectors $a^\mu = (a, \vec{a})$ and $b^\mu = (b, \vec{b})$, with dot product $a \cdot b = -a_0 b_0 + \vec{a}\cdot\vec{b} \equiv a_\mu b^\mu$, where $a_\mu \equiv \eta_{\mu,\nu} a^\mu = (-a_0, \vec{b})$. Here $\eta_{\mu\nu} \equiv diag(-1, 1, 1, 1)$ is the fixed metric of SR.

(GR replaces $\eta_{\mu\nu}$ with a dynamical metric $g_{\mu\nu}(x)$. This will be analogous to A_μ in E&M. The analog of Maxwell equations will be Einstein's equations, relating derivatives of the metric to the "charge" source of gravity: energy and momentum.)

Inertial frames are related by $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$. The dot product is preserved as long as $\eta_{\rho\sigma} = \Lambda^{\mu'}_\rho \Lambda^{\nu'}_\sigma \eta_{\mu'\nu'}$. Examples: rotate in x, y plane $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$; boost along x axis, $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$. Consider the origin $x' = 0$ in the original frame, $x/t = v = \tanh\phi$, so $\sinh\phi = \gamma v$ and $\cosh\phi = \gamma \equiv 1/\sqrt{1-v^2/c^2}$. Set $c = 1$ from now on.

• Continue 4-vectors a^μ, and their inner product $a \cdot b \equiv a^\mu b^\nu \eta_{\mu\nu} \equiv a^\mu b_\mu$. Two inertial frames of reference are related by (taking origins to coincide) $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$. The dot product is preserved as long as

$$\eta_{\rho\sigma} = \Lambda^{\mu'}_\rho \Lambda^{\nu'}_\sigma \eta_{\mu'\nu'}.$$

All Λ satisfying this form the Lorentz group. Note that all such Λ have determinant ± 1, and all those connected to the identity have determinant 1, so they have $d^4x = d^4x'$.

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Heartbeat in $'$ frame: dt', with $dx' = 0$, get $dt = \gamma dt'$, so seems to beat slower (likewise from $ds^2 = -dt^2 + d\vec{x}^2 = -dt'^2$.

Ruler in $'$ frame, length dx'. Measure both ends simultaneously in lab, with $dt = 0$, Then $dx = dx'/\gamma$, length contracted.

Two events are timelike separated if there is a frame where they happen a the same place. In that frame, $\Delta s^2 = \Delta t'^2 \equiv \Delta\tau^2$, where $\Delta\tau$ is the "proper time" between the events. In any other frame, $\Delta t = \gamma\Delta\tau$, time dilation.

For spacelike path, $\Delta s = \int ds = \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$.

For timelike paths, the total proper time is $\Delta\tau = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$. This applies even if there is acceleration. If no acceleration, can write $\Delta\tau = \int \sqrt{1-v^2} dt$.

Consider proper time between timelike separated events A and C. For observer 1, in the frame where they're at the same place, the proper time is $\Delta t = t_C - t_A$. For observer 2, who moves and comes back, the proper time length is $\Delta\tau_{AB'C} = \sqrt{1-v^2}\Delta\tau_{ABC} < \Delta\tau_{ABC}$. Moving twin is younger when they meet again. Non-straight path has shorter proper time. In spacetime, straight path between two events has the longest proper time.

• We saw in earlier lecture that the action is invariant under Gallean transformations provided it is $L = \frac{1}{2} m\vec{v}^2 - U(x)$, linear in v^2. To have physics be the same in all inertial frames, need the equations of motion to be covariant: if satisfied in one frame, they should be satisfied in all others, when observables are properly converted. This is achieved by having the action be a Lorentz scalar.

Free particle action $S = \int L dt$, $L = -mc^2\sqrt{1-v^2/c^2}$. Then $\vec{p} = \partial L/\partial v$ and $H = \vec{p} \cdot \vec{v} - L$ combine into $p^\mu = mu^\mu$. The EOM is then $du^\mu/d\tau = 0$ for a free particle.