1/23/12 Lecture outline

$\star$ See lecture notes for details. Continue where we left off last time.

• Upper vs lower indices, canonical examples: $dx^\mu = (dt, d\vec{x})$ and $\partial_\mu = (\frac{\partial}{\partial t}, \nabla)$, e.g. $\partial_\mu x^\nu = \delta_\mu^\nu$. Under $x^{\mu'} = \Lambda_\nu^{\mu'} x^\nu$, have $dx^{\mu'} = \Lambda_\nu^{\mu'} dx^\nu$ and $\frac{\partial}{\partial x^{\mu'}} = \Lambda_{\mu'}^\nu \frac{\partial}{\partial x^\nu}$, where $\Lambda_{\mu'}^\nu$ is the inverse to $\Lambda_\nu^{\mu'}$. Can write $A^\mu = \frac{\partial x^\mu}{\partial x^{\nu'}} A^{\nu'}$ and $A_\mu = \frac{\partial x^{\nu'}}{\partial x^\mu} A_{\nu'}$.

• Form 4-scalars by contracting all upper and lower indices, e.g $T^{\mu\nu} a_\mu b_\nu$. Can also form 4-scalars using $\epsilon^{\mu\nu\rho\sigma}$; this is used to show $d^4x = d^4x'$, with time-dilation and length contraction factors of γ and $1/\gamma$ canceling.

• 4-velocity, $u^\mu = dx^\mu/d\tau = (\gamma c, \gamma\vec{v})$, so $u^\mu u_\mu = c^2$.

• Free particle action $S = \int L dt$, $L = -mc^2\sqrt{1 - v^2/c^2}$. Then $\vec{p} = \partial L/\partial v$ and $H = \vec{p}\cdot\vec{v} - L$ combine into $p^\mu = (E/c, \vec{p}) = mu^\mu$. The EOM is then $du^\mu/d\tau = 0$ for a free particle.

• Force $f^\mu = \frac{dp^\mu}{d\tau} = (\gamma\frac{dE}{dt}, \gamma\frac{d\vec{p}}{dt})$.

• Addition of velocities, shown from u^μ being a 4-vector.

• 4-acceleration $a^\mu = d^2x^\mu/d\tau^2$, satisfies $a^\mu u_\mu = 0$.

• 4-momentum $p^\mu = (E, \vec{p})$. For massive particle, $p^\mu = mu^\mu$, so $p^\mu p_\mu = (mc)^2$. For a massless object (e.g. photon), we still have $p^\mu = (E, \vec{p})$ as a 4-vector. Here's a way to see that $(E, \vec{p})$ always transforms as a 4-vector. For any theory, the action S must be Lorentz invariant; this ensures that the EOM behave properly under reference frame changes. Now use the fact that energy and momentum can be related to the derivative of the action w.r.t. changes of the endpoint time and position: $L(x_b) - \dot{x}_b(\partial L/\partial\dot{x}_b) = \partial S_{cl}/\partial t_b$, and $\partial L/\partial\dot{x}_b = \partial S_{cl}/\partial x_b$, so we have $p_\mu = \partial S_{cl}/\partial x^\mu$, and the RHS is clearly a 4-vector.

• $k^\mu = (\omega, \vec{k})$, so $e^{ik\cdot x}$ is invariant. On this, $k_\mu = i\partial_\mu$. Fits with QM, where $p^\mu = \hbar k^\mu$. Gives correct Doppler effect relations between $k^{\mu'}$ and k^μ.

• Relativistic kinematics, conservation of total energy and momentum. Examples:

Decay of particle of mass M into particles of mass m_1 and m_2, with energies E_1 and E_2. Show $E_1 = (M^2 + m_1^2 - m_2^2)c^2/2M$.

(NEXT TIME) Pair creation (Jackson 11.22), scatter energetic photon against CMB photon to make an electron-positron pair. What is the minimal energy of the photon? In the CM frame the produced pair is at rest for the minimal energy. So in the frame of the CMB, the two produced particles have the same energy E and momentum p, with $E_1 + E_2 = 2E$ and $p_1 - p_2 = 2p$. End result: $E_1 E_2 = m^2c^4$, where $E_2 = k_BT$ for $T = 3K$.