3/4/13 Lecture outline

• No action at a distance: all interactions are via fields, which transmit the interaction over spacetime. The transmission has a universal maximal speed, $c \approx 2.998 \times 10^{10} cm/sec$.

• Principle of (special) relativity: no local physics experiment can distinguish one inertial frame from another. If one frame is inertial, any moving with constant relative velocity is also inertial.

• Spacetime 4-vector: $x^\mu = (x^0, \vec{x})$, with $x^0 \equiv ct$. Suppose event A is at x^μ_A and event B is at x^μ_B. Let $\Delta x^\mu \equiv x^\mu_B - x^\mu_A = (c\Delta t, \vec{\Delta} x)$ be their spacetime separation. Define the spacetime interval: $\Delta s^2 \equiv (\Delta x^0)^2 - \Delta \vec{x}^2$. If the two events are connected by a light ray, then $\Delta s^2 = 0$, and we say the events are lightlike (or null) separated. If the two events have $\Delta s^2 > 0$, then we say they are time-like separated, e.g. two events at the same position, at differing times. If the two events have $\Delta s^2 < 0$, then we say they are space-like separated, e.g. two events at different locations, at the same time. The statement of no action at a distance means that events A and B can be causally related, e.g. event A is the cause and event B is the effect, only if $\Delta s^2 \geq 0$.

Now consider two inertial frames, the lab and a rocket. The person in the lab uses coordinates $x^\mu = (ct, \vec{x})$ and the person in the rocket uses $x^{\mu'} = (ct', \vec{x}')$. They both observe events A and B. If the events are connected by a light ray, the principle of relativity implies $\Delta s^2 = \Delta s'^2 = 0$. This implies more generally $\Delta s^2 = f(|\vec{v}|)\Delta s'^2$, with $\vec{v}$ the relative velocity of two frames. But inverse transform says $\vec{v} \to -\vec{v}$ must send $f \to 1/f$, so get $f = 1$, i.e. $\Delta s^2 = \Delta s'^2$ is a Lorentz invariant, the same in all inertial frames. Write the invariant interval using $ds^2 \equiv (cdt)^2 - d\vec{x} \cdot d\vec{x}$.

• The Δs^2 between two events is an example of a 4-scalar, a quantity that's the same for all inertial observers. The principle of relativity posits that the result of any experiment is the same in any inertial frame of reference. Relativity says that **every** physical quantity is either a 4-scalar or fits into an appropriate generalization: 4-vector, 4-tensor, that can be used to form frame-invariant physical quantities.

• The action must be a 4-scalar. Then the equations of motion, coming from least action, are guaranteed to be properly related in different frames of reference. (Also, this fits nicely with the path integral description of QM, where $\psi \sim e^{iS/\hbar}$.)

• Another 4-scalar: the total electric charge of an object, or in a "box" in space.

• The mass m of a particle is a 4-scalar.

• Another 4-scalar: the proper time interval between two time-like separated events A and B: $\Delta \tau = \int_A^B d\tau$, where $d\tau^2 \equiv ds^2/c^2$.

• Aside (don't cover in lecture): condition on L for Gallean invariance under $\vec{x}_i = \vec{x}_i' + \vec{v}_0 t$, $\vec{v}_i = \vec{v}_i' + \vec{v}_0$:

$$L(\vec{x}_i, \vec{v}_i, t) = L(\vec{x}_i', \vec{v}_i', t') + \frac{dG}{dt},$$

which implies that L is linear in the $\langle v \rangle_i^2$, $\partial L / \partial \vec{v}_i^2 = \alpha_i$ a constant. Consider $\vec{v}_0$ small, show how it works with $G = \sum_i 2\vec{x}_i \cdot \vec{v}_0 / \alpha_i$. Introduce mass: $\alpha_i = m_i/2$. Writing the EL equations, get forces from gradients of U: infinite signal speed. Not correct: maximum actual signal speed is c (or the speed of superluminal neutrinos .... just kidding).

• More on proper time: read by moving clock. For timelike separated events, there is a frame where the events occur at the same spatial position. The proper time is the time experienced by clocks in that frame. So $d\tau = dt'$ when $d\vec{x}' = 0$.

• Four vectors $a^\mu = (a, \vec{a})$ and $b^\mu = (b, \vec{b})$, with dot product $a \cdot b = a_0 b_0 - \vec{a} \cdot \vec{b} \equiv a_\mu b^\mu$, where $a_\mu \equiv \eta_{\mu\nu} a^\mu = (a_0, -\vec{a})$. Here $\eta_{\mu\nu} \equiv diag(1, -1, -1, -1)$ is the metric of SR[1]. Einstein summation convention: repeated upper and lower indices are always summed over. We saw that $dx^\mu dx_\mu$ is a 4-scalar, the same in all inertial frames. All 4-vectors transform like dx^μ between Lorentz frames. So if a^μ and b^μ are any 4-vectors, then $a_\mu b^\mu$ is a Lorentz invariant scalar.

• Examples of 4-vectors: dx^μ, $p^\mu = (E/c, \vec{p})$, $k^\mu = (\omega/c, \vec{k})$, $J^\mu = (c\rho, \vec{J})$, $A^\mu = (\phi, \vec{A})$, $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla})$. Note that ∂_μ has a lower index, whereas $\partial^\mu = \eta^{\mu\nu} = (\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla})$. The lower index makes sense, for example $\partial_\mu x^\mu = 4$ in all reference frames. The charge conservation equation in this notation is $\partial_\mu J^\mu = 0$. The Lorentz gauge condition is $\partial_\mu A^\mu = 0$. The d'Almbertian is $\partial_\mu \partial^\mu$, so it is a Lorentz invariant scalar. The solution of the wave equation $\partial^2 \phi$, e.g. $\phi = Ae^{i\vec{k}\cdot\vec{x} - \omega t}$ is Lorentz invariant, since $k_\mu x^\mu = \omega t - \vec{k} \cdot \vec{x}$.

• Inertial frames are related by a **linear** relation: $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$. All 4-vectors transform the same way, with the same $\Lambda_\nu^{\mu'}$, i.e. : $a^{\mu'} = \Lambda_\nu^{\mu'} a^\nu$, and $b^{\mu'} = \Lambda_\nu^{\mu'} b^\nu$. The relativistic dot product of two 4-vectors is a 4-scalar, i.e. the same in all frames: $a_\mu b^\mu = a_{\mu'} b_{\mu'}$. This condition determines the allowed transformations: the dot product is preserved as long as

$$\eta_{\rho\sigma} = \Lambda_\rho^{\mu'} \Lambda_\sigma^{\nu'} \eta_{\mu'\nu'}.$$

All Λ satisfying this form the Lorentz group. Note that all such Λ have determinant ± 1, and all those connected to the identity have determinant 1, so they have $d^4x = d^4x'$.

[1] Aside (don't cover in lec.): GR replaces $\eta_{\mu\nu}$ with a dynamical metric $g_{\mu\nu}(x)$. This will be analogous to A_μ in E&M. The analog of Maxwell equations will be Einstein's equations, relating derivatives of the metric to the "charge" source of gravity: energy and momentum.

Examples: rotate in x, y plane $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$; boost along x axis, $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$. Consider the origin $x' = 0$ in the original frame, $x/t = v = \tanh\phi$, so $\sinh\phi = \gamma v$ and $\cosh\phi = \gamma \equiv 1/\sqrt{1 - v^2/c^2}$. Note taking $v \to -v$ gives the inverse transformation. Will often set $c = 1$.

Heartbeat in $'$ frame: dt', with $dx' = 0$, get $dt = \gamma dt'$, so seems to beat slower (likewise from $ds^2 = -dt^2 + d\vec{x}^2 = -dt'^2$.

Ruler in $'$ frame, length dx'. Measure both ends simultaneously in lab, with $dt = 0$, Then $dx = dx'/\gamma$, length contracted.

Two events are timelike separated if there is a frame where they happen a the same place. In that frame, $\Delta s^2 = \Delta t'^2 \equiv \Delta\tau^2$, where $\Delta\tau$ is the "proper time" between the events. In any other frame, $\Delta t = \gamma \Delta\tau$, time dilation.

For spacelike path, $\Delta s = \int ds = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$. For timelike paths, the total proper time is $\Delta\tau = \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$. This applies even if there is acceleration[2]. If no acceleration, can write $\Delta\tau = \int \sqrt{1 - v^2} dt$.

• Since dx^μ is a 4-vector and $d\tau$ is a scalar, we can form another 4-vector, the 4-velocity: 4-velocity, $u^\mu = dx^\mu/d\tau = \gamma \frac{dx^\mu}{dt} = (\gamma c, \gamma \vec{v})$, so $u^\mu u_\mu = c^2$.

For a free particle of mass m, $p^\mu = m u^\mu$.

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[2] Aside (don't cover): Consider proper time between timelike separated events A and C. For observer 1, in the frame where they're at the same place, the proper time is $\Delta t = t_C - t_A$. For observer 2, who moves and comes back, the proper time length is $\Delta\tau_{AB'C} = \sqrt{1 - v^2} \Delta\tau_{ABC} < \Delta\tau_{ABC}$. Moving twin is younger when they meet again. Non-straight path has shorter proper time. In spacetime, straight path between two events has the *longest* proper time.