3/6/13 Lecture outline

• Last time: Lorentz transformation between frames, $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$. All 4-vectors transform the same way, with the same $\Lambda^{\mu'}_\nu$. Recall boost along the x axis: $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$, with $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$. Inverse transformation $= \beta \to -\beta$.

• Examples of 4-vectors: $x^\mu = (ct, \vec{x})$, $p^\mu = (E/c, \vec{p})$, $J^\mu = (c\rho, \vec{J})$, $A^\mu = (\phi, \vec{A})$, $u^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = \gamma(c, \vec{v})$.

• Example application: Find $\vec{\phi}$ and $\vec{A}$ of a particle of charge q, moving with velocity v along the x axis. We worked this out, the hard way, directly from Maxwell's equations. Now let's see it as an immediate consequence of relativity. In the rocket frame moving with the particle, we have $A^{\mu'} = (\phi', \vec{A}') = (q/r', \vec{0})$. Converting to the lab frame,

$$\begin{pmatrix} \phi \\ A_x \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} q/r' \\ 0 \end{pmatrix},$$

which gives the answer we found earlier, since $r' = \sqrt{x'^2 + y'^2 + z'^2}$ and $x' = \gamma(x - vt)$.

• We discussed last week the relativistic Lagrangian for a mass m particle of charge q, interacting with $\vec{E}$ and $\vec{B}$:

$$L = -mc^2\sqrt{1 - v^2/c^2} + \frac{q}{c}\vec{v} \cdot \vec{A} - q\phi.$$

Now we can understand why it gives a Lorentz invariant action, since this $S = \int dt L$ can be written as a manifestly Lorentz invariant integral over the particle's world-line, $x^\mu(\tau)$:

$$S = \int (-mc^2 d\tau - \frac{q}{c} A_\mu dx^\mu)$$

We saw last week that the above L gives Lorentz force law as its equations of motion:

$$\frac{d}{dt}(\gamma m \vec{v}) = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B}.$$

• We're guaranteed that the above force law is relativistic, since it came from a relativistic action. But the action involves the 4-vector $A^\mu = (\phi, \vec{A})$. Let's now discuss the Lorentz transformation properties of $\vec{E}$ and $\vec{B}$. They fit in $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Write out the components in terms of $\vec{E}$ and $\vec{B}$. Likewise for $F_{\mu\nu}$.

• If $x^{\mu'} = \Lambda^{\mu'}_\nu x^\nu$, then a two-index tensor $A^{\mu\nu}$, e.g. like $F^{\mu\nu}$, transforms as $A^{\mu'\nu'} = \Lambda^{\mu'}_\rho \Lambda^{\nu'}_\sigma A^{\rho\sigma}$. Example using boost along the x axis, transforming $F^{\mu\nu}$ and read off transformation of $\vec{E}$ and $\vec{B}$. Get $E_x = E'_x$, $B_x = B'_x$,

$$\begin{pmatrix} E_y \\ B_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E_y \\ B_z \end{pmatrix}, \qquad \begin{pmatrix} E_z \\ B_y \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E'_z \\ B'_y \end{pmatrix}.$$

• If p^μ is a 4-vector, we can define a force 4-vector $f^\mu = \frac{dp^\mu}{d\tau} = \gamma\frac{dp^\mu}{dt}$. So the spatial part of the Lorentz force law can be written as

$$\frac{dp^\mu}{d\tau} = f^\mu = \gamma\frac{dp^\mu}{dt} = \gamma(q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}) = \frac{q}{c}F^{\mu\nu}u_\nu.$$

The time component gives the power: $\gamma\frac{d\mathcal{E}}{dt}$.

• Maxwell's equations can now be written as 4-vector equations: $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J_\nu$. The no-magnetic source Maxwell equations can be written as $\partial_\mu \widetilde{F}^{\mu\nu}$, where $\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, or equivalently $\partial_\mu F_{\rho\sigma}+\partial_\rho F_{\sigma\mu}+\partial_\sigma F_{\mu\rho} = 0$; we solved these already, via $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

As we already saw, Maxwell's equation requires charge conservation, which is now obvious from summing over the indices, since $\partial_\mu\partial_\nu$ is symmetric and $F^{\mu\nu}$ is antisymmetric: $0 = \partial_\mu\partial_\nu F^{\mu\nu} = \frac{4\pi}{c}\partial^\nu J_\nu$.

• Moving point charge: $J^\mu = c\rho\frac{dx^\mu}{dx^0}$, which is a 4-vector because ρ and dx^0 transform the same way. Likewise, $\rho = q\delta^3(\vec{x}-\vec{x}_0)$ makes sense, with q Lorentz invariant. E.g. $\delta^4(x^\mu - x_0^\mu)$ is Lorentz invariant, and $\delta(t-t_0)dt$ is Lorentz invariant

• Using the above, the term $-\frac{q}{c}A_\mu dx^\mu$ in the point particle world-line action can be written as a spacetime volume integral $-\int d^4x A_\mu J^\mu$, which is Lorentz invariant. Note d^4x is Lorentz invariant and $\epsilon^{\mu\nu\rho\sigma}$ is Lorentz invariant for the same reason, mentioned last time: $\det\Lambda = 1$.

Next time: write Maxwell's equations as coming from least action, with Lagrangian density $\sim F_{\mu\nu}F^{\mu\nu} \sim \vec{E}^2 - \vec{B}^2$.