3/11/13 Lecture outline

• Recall $A^\mu = (\phi, \vec{A})$, $j^\mu = (c\rho, \vec{j})$. Lorentz force law:

$$\frac{dp^\mu}{d\tau} = f^\mu = \gamma\frac{dp^\mu}{dt} = \gamma(q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}) = \frac{q}{c}F^{\mu\nu}u_\nu.$$

The time component gives the power: $\gamma\frac{d\mathcal{E}}{dt}$. Maxwell's equations: $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu$ and $\partial_\mu \widetilde{F}^{\mu\nu}$, where $\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, or equivalently $\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho} = 0$; we solve the latter via $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Using transformation $F_{\mu'\nu'} = \Lambda_\mu^{\mu'}\Lambda_\nu^{\nu'}F^{\mu\nu}$, find

$$\vec{E}_{||} = \vec{E}'_{||}, \qquad \vec{E}_\perp = \gamma(\vec{E}' - \frac{\vec{v}}{c}\times\vec{B}')_\perp$$

$$\vec{B}_{||} = \vec{B}'_{||}, \qquad \vec{B}_\perp = \gamma(\vec{B}' + \frac{\vec{v}}{c}\times\vec{E}')_\perp$$

• Moving point charge: $J^\mu = c\rho\frac{dx^\mu}{dx^0}$, which is a 4-vector because ρ and dx^0 transform the same way. Likewise, $\rho = q\delta^3(\vec{x} - \vec{x}_0)$ makes sense, with q Lorentz invariant. E.g. $\delta^4(x^\mu - x_0^\mu)$ is Lorentz invariant, and $\delta(t - t_0)dt$ is Lorentz invariant. The term $-\frac{q}{c}A_\mu dx^\mu$ in the point particle world-line action can thus be written as a spacetime volume integral $-\frac{1}{c}\int d^4x A_\mu J^\mu$, which is Lorentz invariant. Note d^4x is Lorentz invariant and $\epsilon^{\mu\nu\rho\sigma}$ is Lorentz invariant for the same reason, mentioned last time: $\det\Lambda = 1$.

• Lorentz invariants: $F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2)$ and $F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\vec{E}\cdot\vec{B}$. Note that if $\vec{E}\cdot\vec{B} = 0$, then there is a frame where the field is entirely $\vec{E}'$ or entirely $\vec{B}'$, depending on the sign of $F_{\mu\nu}F^{\mu\nu}$.

• Maxwell's equations as a field theory:

$$S_{field} = \int d^4x\mathcal{L}_{field}, \qquad \mathcal{L}_{field} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$$

where we impose $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The field interacts with charges via

$$S_{int} = -\frac{1}{c}\int d^4x A_\mu J^\mu.$$

Varying $A^\mu \to A^\mu + \delta A^\mu$ and requiring that the action be stationary gives Maxwell's equations.

• The total action is $S = S_{field} + S_{matter} + S_{int}$. Spacetime translation symmetry, $x^\mu \to x^\mu + \epsilon^\mu$ is related to conservation of $P^\mu = (H, c\vec{P})$. We saw that electric charge conservation is equivalent to $\partial_\mu J^\mu = 0$. Likewise, conservation of P^μ is

$$P^\mu = \int d^3x T^{\mu 0} \quad \text{conserved} \quad \leftrightarrow \quad \partial_\nu T^{\mu\nu} = 0.$$

The relation between the conservation law and the symmetry is Noether's theorem:

$$\frac{d}{dx_\mu}\mathcal{L} = \frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\lambda)}\partial^\mu A_\lambda\right) + \frac{\partial \mathcal{L}}{\partial x_\mu}$$

which implies

$$\partial_\nu T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial x_\mu}, \qquad T^{\mu\nu}_{field} = \frac{\partial \mathcal{L}_{field}}{\partial(\partial_\nu A_\lambda)}\partial^\mu A_\lambda - g^{\mu\nu}\mathcal{L}_{field}.$$

So if there is no explicit x_μ dependence then the conservation equation for $T^{\mu\nu}$ is satisfied. We'll show next time how this gives the field contribution to energy and momentum densities, and how $T^{\mu\nu}_{tot} = T^{\mu\nu}_{matter} + T^{\mu\nu}_{field}$, with (only) the sum conserved.