3/13/13 Lecture outline

• Last time: $S = S_{matter} + S_{field} + S_{int}$, where A^μ appears in

$$\mathcal{L}_{field} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}, \qquad \mathcal{L}_{int} = -\frac{1}{c} A_\mu J^\mu.$$

We discussed how spacetime translation symmetry, $x^\mu \to x^\mu + \epsilon^\mu$ is related to conservation of $P^\mu = (H, c\vec{P})$, which are the conserved "charges" associated with the locally conserved "currents," the stress-energy tensor:

$$P^\mu = \int d^3x T^{\mu 0} \quad \text{conserved} \quad \leftrightarrow \quad \partial_\nu T^{\mu\nu} = 0.$$

As we discussed, the relation between the conservation law and the symmetry is Noether's theorem:

$$\frac{d}{dx_\mu}\mathcal{L} = \frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\lambda)}\partial^\mu A_\lambda\right) + \frac{\partial \mathcal{L}}{\partial x_\mu}$$

which implies

$$\partial_\nu T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial x_\mu}, \qquad T^{\mu\nu}_{field} = \frac{\partial \mathcal{L}_{field}}{\partial(\partial_\nu A_\lambda)}\partial^\mu A_\lambda - g^{\mu\nu}\mathcal{L}_{field}.$$

• Time out, for a bit more detail about the stress-energy (also called the energy-momentum) tensor. The amount of energy and momentum in an volume V is given by:

$$P^\mu = \int_V d^3x T^{\mu 0}.$$

So the time derivative is

$$\frac{d}{dx^0}P^\mu(x_0) = \int_V d^3x \partial_0 T^{\mu 0} = -\int_V d^3x \partial_i T^{\mu i} = -\int_{\partial V} T^{\mu i} da^i$$

where da^i is the area element pointing along the outward normal. So $T^{0i} = S^i$ is the Poynting vector, the energy flux. Likewise, T^{ij} is the force per area, that we studied before. Recall that the electromagnetic force on the charges in a volume V is given by the Lorentz force law:

$$\frac{d}{dt}\vec{P}_{matter} = \int_V d^3x(\rho\vec{E} + c^{-1}\vec{J}\times\vec{B})$$

Recall also that the field momentum in the volume V has

$$\frac{d}{dt}\vec{P}_{field} = \int_V d^3x \frac{\partial}{\partial t}\frac{\vec{E}\times\vec{B}}{4\pi c}.$$

As we discussed before, the conservation of total $\vec{P}_{tot} = \vec{P}_{field} + \vec{P}_{matter}$ follows from Maxwell's equation,

$$0 = \rho \vec{E} + \frac{1}{c}\vec{J} \times \vec{B} + \frac{\partial}{\partial ct}\vec{S}/c + \partial_i T^{ij}_{field}\hat{e^j}.$$

• OK, back to where we left off. Obtain

$$T^{\mu\nu}_{field} = -\frac{1}{4\pi}F^{\mu\lambda}F^{\nu}_{\lambda} + \frac{1}{16\pi}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}.$$

Aside: we added an improvement term $\partial_\lambda \psi^{\nu\nu\lambda}$ with $\psi^{\mu\nu\lambda} = \frac{1}{4\pi}F^{\mu\lambda}A^{\nu}$, which was needed to make the stress tensor properly symmetric (needed for conservation of angular momentum, using $M^{\mu\nu\lambda} = x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}$. The improvement term is also needed to make $T^{\mu\nu}_{field}$ properly gauge invariant. Verify that the components agree with what we found before.

Using Maxwell's equations, we find

$$\partial_\mu T^{\mu\nu}_{field} = -\frac{1}{c}F^{\nu\lambda}J_\lambda.$$

For the matter part, using the Lorentz force law, we'll show that

$$\partial_\mu T^{\mu\nu}_{matter} = +\frac{1}{c}F^{\nu\lambda}J_\lambda.$$

For example, we recognize $\frac{\partial}{\partial t}\mathcal{E}_{matter} = \vec{J}\cdot\vec{E}$. So the total $T^{\mu\nu}_{tot} = T^{\mu\nu}_{field} + T^{\mu\nu}_{matter}$ is conserved, $\partial_\mu T^{\mu\nu} = 0$. Let's discuss the matter part more, treating it as a collection of point particles of mass m_n, at positions x^μ_n. Then $T^{\mu 0} = \sum_n c p^\mu_n \delta^3(\vec{x} - \vec{x}_n(t))$, so

$$T^{\mu\nu}_{matter} = \sum_n p^\mu_n \frac{dx^\nu_n}{dt}\delta^3(\vec{x} - \vec{x}_n(t)) = \sum_n c^2 \frac{p^\mu_n p^\nu_n}{E_n}\delta^3(\vec{x} - \vec{x}_n(t)).$$

(Aside: $c^2 T^{\mu\nu}_{matter} = \epsilon u^\mu u^\nu$, where ϵ is the energy density, $\epsilon = \sum m_n c^2 \gamma^{-1}\delta^3(\vec{x} - \vec{x}_n)$. Then $\partial_\mu T^{\mu\nu}_{matter} = \epsilon u^\mu \partial_\mu u^\nu$, since matter conservation gives $\partial_\mu(\epsilon u^\mu) = 0$.) Note that

$$\frac{\partial}{\partial x^i}T^{i\nu}_{matter} = \sum_n p^\nu_n \frac{dx^i_n}{dt}(-\frac{\partial}{\partial x^i_n})\delta^3(\vec{x} - \vec{x}_n(t)) = \sum_n p^\nu_n(-\frac{\partial}{\partial t})\delta^3(\vec{x} - \vec{x}_n(t))$$
$$= -\frac{\partial}{\partial x^0}T^{0\nu}_{matter} + \sum_n(\frac{d}{dt}p^\nu_n(t))\delta^3(\vec{x} - \vec{x}_n(t)),$$

So

$$\partial_\mu T^{\mu\nu}_{matter} = \sum_n \frac{dp^\nu_n}{dt}\delta^3(\vec{x} - \vec{x}_n(t)) = \frac{1}{c}F^{\nu\lambda}\sum_n q_n \frac{dx^\lambda_n}{dt}\delta^3(\vec{x} - \vec{x}_n(t)) = \frac{1}{c}F^{\nu\lambda}J_\lambda$$

To summarize, we see how $T^{\mu\nu}_{tot}$ is conserved, and how the field contribution can be understood directly from the Lagrangian and Noether's method.

• Aside: in the rest frame of a fluid, $T^{\mu\nu} = diag(\epsilon, p, p, p)$, where ϵ is the energy density and p is the pressure. The relativistic expression in a general frame is then $T^{\mu\nu} = (\epsilon + p)u - pg^{\mu\nu}$. Note that $T^\mu_\mu = \epsilon - 3p$ and one can show a viral theorem $\epsilon - 3p = \sum_n m_n c^2\sqrt{1 - v_n^2/c^2}$. For massless particles, $\epsilon = 3p$. For vacuum energy density (cosmological constant), $T^{\mu\nu} \sim g^{\mu\nu}$, so $\epsilon = -p$.