1/9/13 Lecture outline

⋆ Garg chapter 3.

• Electrostatics: set all $\partial_t \to 0$ in Maxwell's equations. $\rho = \rho(\vec{r})$ (no t) and $\vec{J} = 0$. Gives $\vec{E} = \vec{E}(\vec{r})$ and $\vec{B} = 0$. Force on a charge q is $\vec{F} = q\vec{E}$, where $\vec{E}$ comes from all the other charges.

• $\nabla \cdot \vec{E} = 4\pi\rho$, $\nabla \times \vec{E} = 0$, $\vec{E} = -\nabla\phi$. Integrating,

$$\int_V dV\, \nabla \cdot \vec{E} = -\int_V dV\, \nabla^2\phi = \int_{\partial V} \vec{E} \cdot d\vec{a} = 4\pi Q_{encl}.$$

• Point charge Q at the origin, $\vec{E}(\vec{r}) = Q\hat{r}/r^2 = Q\vec{r}/r^3$. This satisfies $\nabla \cdot \vec{E} = 4\pi\rho = 4\pi Q\delta(x)\delta(y)\delta(z) = r^{-2}Q\delta(r)$.

Get general case by superposition, translation symmetry:

$$\vec{E}(\vec{x}) = \sum_n q_n \frac{\vec{x} - \vec{x}_n}{|\vec{x} - \vec{x}_n|^3} = \int d^3\vec{x}'\rho(\vec{x}')\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3},$$

with $\rho(\vec{x}') = \sum_n q_n\delta(\vec{x}' - \vec{x}_n)$. You can verify that this $\vec{E}$ has vanishing curl. Therefore, it can be written as the gradient of a scalar function. Indeed,

$$\vec{E} = -\nabla\phi, \qquad \phi = \sum_n \frac{q_n}{|\vec{x} - \vec{x}_n|} = \int d^3\vec{x}'\frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}.$$

Nice: it's easier to work with a scalar than a vector. We can also consider surface charges $\rho = \sigma\delta(z)$ where z is the normal coordinate to the surface, or line charges $\rho = \lambda\delta(x)\delta(y)$, e.g. $\phi(\vec{x}) = \int da'\sigma(\vec{x}')/|\vec{x} - \vec{x}'|$ or $\phi(\vec{x}) = \int d\ell'\lambda(\vec{x}')/|\vec{x} - \vec{x}'|$. Surface charges lead to discontinuity in $\vec{E}$, $(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = 4\pi\sigma$.

• In cases with enough symmetry, we can bypass the above, in favor of using Gauss' law, $\oint_{\partial V} \vec{E} \cdot d\vec{a} = 4\pi Q_{encl}$ to directly compute $\vec{E}$. Examples: uniform spherical charge distribution (inside and out), uniform charged thick plate, uniform charged cylinder. Can get simple, variants by superposition. E.g. cylinder with some regions cut out.

• Work done in moving charge q from $\vec{r}_1$ to $\vec{r}_2$ is

$$W = -\int_1^2 \vec{F} \cdot d\vec{r} = q(\phi(\vec{r}_2) - \phi(\vec{r}_1)).$$

Assemble a collection of charges by brining them in from infinity. The potential energy is then found to be

$$U = \tfrac{1}{2}\sum_{n \neq m} \frac{q_n q_m}{|\vec{x}_n - \vec{x}_m|} = \tfrac{1}{2}\sum_n q_n\phi(\vec{x}_n) = \tfrac{1}{2}\int d^3\vec{x}\rho(x)\phi(\vec{x}),$$

where the $\frac{1}{2}$ is to compensate for double counting pairs.

• Gauss' law. Consider a charge Q at the origin, so $\vec{E} = Q\hat{r}/r^2$. Note that $\int_{\partial V} \vec{E}\cdot\vec{da} = 4\pi Q = \int_V d^3\vec{r}\,\nabla\cdot\vec{E}$. So $-\nabla^2\phi$ here integrates to 4π. Indeed, $\nabla^2(1/r) = -4\pi\delta(\vec{r})$. More generally, we can see that the above is compatible with $\nabla\cdot\vec{E} = -\nabla^2\phi = 4\pi\rho$ because

$$\nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\delta(\vec{r}-\vec{r}').$$

• The above is an example of solving a differential equation, with source term, via a Green's function. We want to solve Poisson's equation, $\nabla^2\phi = -4\pi\rho$ for ϕ, subject to some boundary conditions. The general solution is a sum of a homogeneous and particular solution. A way to phrase this

$$\phi(\vec{x}) = \int d^3\vec{x}' G(\vec{x},\vec{x}')\rho(\vec{x}'), \qquad \text{where} \qquad \nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}'),$$

where $G(\vec{x},\vec{x}')$ is the Green's function with appropriate boundary conditions. We'll have fun later, when we study conductors, with various setups and possible boundary conditions. Let's just for now make some general remarks.

Suppose that we are given ρ and are asked to solve the differential equation $\nabla^2\phi = -4\pi\rho$ for ϕ. There are some theorems about the uniqueness of the solutions. Consider Green's identities:

$$\int_V d^3x(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi) = \oint_{\partial V} da\ \phi\frac{\partial\psi}{\partial n}$$
$$\int_V d^3x(\phi\nabla^2\psi - \psi\nabla^2\phi) = \oint_{\partial V} (\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n})da.$$

Apply the first to $\phi = \psi = \phi_1 - \phi_2 \equiv X$, the difference between two solutions of Poisson's equation, implies that $\int_V \nabla X^2 = \int_{\partial V} X\frac{\partial X}{\partial n}da$. Impose D (Dirichlet) or N (Neumann) BCs on the surface, implies $X = 0$ or $\partial_n X = 0$, respectively. Then get $\nabla X = 0$ in the interior, which, using the BCs, implies that $X = 0$. Generally, specifying D or N fully specifies the solution. Specifying both is generally inconsistent (or quantizes the modes of the solutions). In other words, $G(\vec{x},\vec{x}') = |\vec{x}-\vec{x}'|^{-1} + F(\vec{x},\vec{x}')$, where F is fully specified by the D or N BCs.

For now, we're working in the vacuum. So the BC is that the fields die off sufficiently fast at infinity, so that $X\partial_r X \to 0$ faster than $1/r^2$.

• Using the above,

$$U = -\frac{1}{8\pi}\int_V d^3x\phi\nabla^2\phi = \frac{1}{8\pi}\int d^3x\vec{E}^2 + \frac{1}{8\pi}\oint_{\partial V}\phi\vec{E}\cdot da.$$

Taking the size of $V \to \infty$, the fields fall off fast enough so the second term vanishes.

• Multipole expansion. Charges at locations $\vec{r}'$ and observer at location $\vec{r}$. Suppose the charges are localized and the observer is far away, $r \gg r'$. Then Taylor expand:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} - (\vec{r}' \cdot \nabla)\frac{1}{r} + \frac{1}{2!}(\vec{r}' \cdot \nabla)^2 \frac{1}{r} + \dots .$$

So then get for large r (far from charged object)

$$\phi(\vec{r}) = \int_V d^3\vec{r}' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \approx \frac{Q}{r} + \frac{d \cdot \vec{r}}{\vec{r}^3} + \tfrac{1}{2}Q_{ij}(3x_i x_i - \delta_{ij} r^2) r^{-5} + \dots,$$

where $Q = \int d^3x' \rho(\vec{x}')$ is the total charge and the dipole and quadrapole moments are

$$\vec{d} = \int d^3\vec{x}' \rho(\vec{x}')\vec{x}', \qquad Q_{ij} = \int d^3x' \rho(\vec{x}')(x_i' x_j' - \frac{1}{3} r'^2 \delta_{ij}),$$

and we used $\partial_i \partial_j 1/r = (3x_i x_j - \delta_{ij} r^2)/r^5$. The dipole part of the electric field is

$$\vec{E}_{\vec{d}} = -\nabla \frac{d \cdot \vec{r}}{r^3} = \frac{3\hat{r} \cdot \vec{d}\hat{r} - \vec{d}}{r^3}.$$

Draw picture.

• More generally, can use

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\hat{r}') Y_{\ell m}(\hat{r})$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\hat{r} \cdot \hat{r}').$$

• Expand the potential energy for a system of charges (say at the origin) in some external electric field: $U \approx Q\phi(0) - \vec{d} \cdot \vec{E}(0) + \dots$. This means that the dipole feels a torque from the external field, which fits with $\vec{\tau} = \sum_n \vec{r}_n \times q_n \vec{E} = \vec{d} \times \vec{E}$.

A dipole dipole interaction is $U_{dd} = (\vec{d}_1 \cdot \vec{d}_2 - 3\vec{d}_1 \cdot \hat{r}\vec{d}_2 \cdot \hat{r})/r^3$. Minimized for dipoles that are parallel to each other, and their separation – lined up head-to head.