1/14/13 Lecture outline

$\star$ Garg chapter 4.

• Finish up from electrostatics: the multiple expansion dipole term $\phi^{(1)} = \vec{d}\cdot\vec{r}/r^3$ leads to

$$\vec{E}_{\vec{d}} = -\nabla\frac{d\cdot\vec{r}}{r^3} = \frac{3(\hat{r}\cdot\vec{d})\hat{r}-\vec{d}}{r^3}.$$

More generally, can use

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y^*_{\ell m}(\hat{r}')Y_{\ell m}(\hat{r})$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\hat{r}\cdot\hat{r}').$$

Expand the potential energy for a system of charges (say at the origin) in some external electric field: $U\approx Q\phi(0)-\vec{d}\cdot\vec{E}(0)+\ldots$. This means that the dipole feels a torque from the external field, which fits with $\vec{\tau}=\sum_n \vec{r}_n\times q_n\vec{E}=\vec{d}\times\vec{E}$.

A dipole dipole interaction is $U_{dd}=(\vec{d}_1\cdot\vec{d}_2-3\vec{d}_1\cdot\hat{r}\vec{d}_2\cdot\hat{r})/r^3$. Minimized for dipoles that are parallel to each other, and their separation – lined up head-to head.

• Segue into magnetic statics. No magnetic monopoles (yet). Magnetic dipoles, analogous to electric dipoles. Consider magnetic dipoles, $\vec{m}_1$ and $\vec{m}_2$, located at $\vec{r}_1$ and $\vec{r}_2$. Then

$$U(\vec{r}_{21}) = \frac{\vec{m}_1\cdot\vec{m}_2 r_{21}^2 - 3(\vec{m}_1\cdot\vec{r}_{21})(\vec{m}_2\cdot\vec{r}_{21})}{r_{21}^5.}$$

(In SI units, an extra factor of $\mu_0/4\pi$.) This is associated with $\vec{B}$: a test magnetic dipole $\vec{m}$ in a magnetic field has $U=-\vec{m}\cdot\vec{B}$, which can be measured via the associated torque $\vec{\tau}=\vec{m}\times\vec{B}$ and force $\vec{F}=\nabla(\vec{m}\cdot\vec{B})$. So the magnetic dipole has a $\vec{B}$ given by

$$\vec{B} = \frac{3(\hat{r}\cdot\vec{m})\hat{r}-\vec{m}}{r^3}.$$

We can write this as either $\vec{B}=\nabla\times\vec{A}$ or as $\vec{B}=-\nabla\phi_{mag}$,

$$\vec{A}=\frac{\vec{m}\times\vec{r}}{r^3}, \qquad \phi_{mag}=\frac{\vec{m}\cdot\vec{r}}{r^3}.$$

Generally, ϕ_{mag} exists only regions where $\vec{J}=0$, and has discontinuities across regions with surface currents (see below).

• Note that $\vec{m}$ is an axial vector. As we'll see in a sec, it is created by charges with angular momentum. Indeed, $\vec{m} = q\vec{L}/2Mc$ at the classical level. For quantum spins, $\vec{m} = gq\vec{S}/2Mc$, where it follows from the Dirac equation that $g \approx 2$ (and, e.g. for the electron, the higher order corrections can be computed to fantastic accuracy in quantum electrodynamics and compared with experiment. Most accurately tested non-trivial prediction ever, in all of science).

• Fit this now with $\vec{F} = q\frac{\vec{v}}{c} \times \vec{B}$. Consider a wire element with current I, and note $d\vec{F} = Id\vec{\ell}\times\vec{B}/c$. Considering a small loop, can obtain $\tau = \vec{m}_{loop}\times\vec{B}$ and $\vec{F} = \nabla(\vec{m}_{loop}\cdot\vec{B})$, with $\vec{m}_{loop} \equiv Id\vec{a}$. Add up the $\vec{B}$ contributions due to many $\vec{m}_{loop}$ contributions, get an integral over an area, and use Stoke's result, writing $\vec{B}$ in terms of $\vec{m} = \frac{I}{2c}\oint \vec{x}\times d\vec{\ell}$, which indeed has $|\vec{m}| = IA/c$.

Obtain the formula of Biot and Savart

$$\vec{B}(\vec{r}) = \frac{I}{c}\oint \frac{d\vec{\ell}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}.$$

We can now show that this has $\nabla \cdot \vec{B} = 0$ and $\oint \vec{B}\cdot d\vec{\ell} = 4\pi I_{encl}/c$.