1/23/13 Lecture outline

⋆ Finish Garg chapter 4, start chapter 5.

• Last time, magnetostatics:

$$\vec{A}(\vec{x}) = \frac{1}{c}\int d^3\vec{x}'\,\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|},$$

$$\vec{B}(\vec{r}) = \frac{I}{c}\oint \frac{d\vec{\ell}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = \frac{1}{c}\int d^3\vec{x}'\,\frac{\vec{j}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}|^3}$$

E.g. for current loop, field on axis is $B_z = 2\pi R^2 I/(c(R^2+z^2)^{3/2}$.

• Recall, $\nabla\cdot\vec{E} = 4\pi\rho \to (\vec{E}_1-\vec{E}_2)\cdot\hat{n} = 4\pi\sigma$. Likewise, $\nabla\times\vec{B} = 4\pi\vec{J}/c \to \oint\vec{B}\cdot d\vec{\ell} = 4\pi I_{encl}/c \to \hat{n}\times(\vec{B}_1-\vec{B}_2) = 4\pi\vec{K}/c$. Also, $(\vec{B}_1-\vec{B}_2)\cdot\hat{n}=0$. E.g. infinite current sheet.

• Standard example (qual, often): a hollow spherical shell of radius a and uniform charge density σ is spinning with angular velocity ω. Find $\vec{B}$ and $\vec{A}$ everywhere. Solution:

$$\vec{A}(\vec{x}) = \int \sigma a^2 d\Omega'\,\frac{\omega a\hat{z}\times\vec{x}'}{c|\vec{x}-\vec{x}'|}$$

since $\vec{K} = \sigma\vec{v} = \sigma\vec{\omega}\times a\hat{r}$. Evaluate the integral using the spherical harmonic expansion of $1/|\vec{x}-\vec{x}'|$, noting that the integral projects to $\ell=1$. Get

$$\vec{A}(\vec{r}) = \frac{\vec{m}\times\vec{r}}{r^3}\quad(r>a),\qquad \vec{A}(\vec{r}) = \frac{\vec{m}\times\vec{r}}{a^3}\quad(r<a).$$

$\vec{m} = \frac{4\pi}{3}\frac{\vec{\omega}\sigma a^4}{c}$. Find $\vec{B}$ outside is a magnetic dipole, and $\vec{B}$ inside is a constant.

• Magnetic scalar potential: in regions where $\vec{J}=0$, can write $\vec{B} = -\nabla\phi_{mag}$, with $\phi_{mag} = \int d^3\vec{r}'\vec{M}\cdot\nabla\frac{1}{|\vec{r}-\vec{r}'|}$. For the above example, get $\vec{M} = \vec{m}/(4\pi a^3/3)$.

Faster: It is clear from the symmetry that this has $\ell=1$ only, from the vector $\vec{\omega}$ source. So $\phi_m = C\cos\theta/r^2$ outside and $\phi_m = -Dr\cos\theta = -Dz$ inside. This gives, with $\vec{m}\equiv C\hat{z}$, and $\vec{D} = D\hat{z}$,

$$\vec{B}_{out} = \frac{3(\hat{r}\cdot\vec{m})\hat{r}-\vec{m}}{r^3},\qquad \vec{B}_{in} = \vec{D}.$$

Impose $\vec{B}\hat{r}$ must be continuous at the surface, so $\vec{D} = 2\vec{m}/a^3$. At the surface, $\hat{r}\times(\vec{B}_{out}-\vec{B}_{in}) = -3\hat{r}\times\vec{m}/a^3 = 4\pi\vec{K}/c$, which determines $\vec{m}$, giving the same answer as above.

Note: right answer comes from imposing continuity of $\partial_r\phi_{mag}$. If we instead impose continuity of ϕ_{mag} would give the wrong answer, $\vec{B}_{in}^{wrong} = -\vec{m}/a^3 = B_{in}^{right} - 3\vec{m}/a^3$.

Recall $\vec{B}^{wrong} - \vec{B}^{right} = -4\pi\vec{M}$, as we saw last time for the point dipole case $\vec{M} = \vec{m}\delta(\vec{r})$. Will come back to this later, with magnetized materials.

• Recall $I_{SI} = \sqrt{4\pi\epsilon_0} I_{Gau}$ and $\vec{B}_{SI} = \sqrt{\frac{\mu_0}{4\pi}}\vec{B}_{Gau}$ (likewise for $\vec{A}$). So $\vec{m}_{SI} = \sqrt{\frac{4\pi}{\mu_0}}\vec{m}_{Gau}$, to have $U = -\vec{m}\cdot\vec{B}$ the same. The SI unit of $\vec{B}$ is the tesla, while the Gaussian unit is the gauss, with 1 tesla = 10^4 gauss.

• Magnetic flux $\Phi = \int_S d\vec{a}\cdot\vec{B} = \oint_{\partial S}\vec{A}\cdot d\vec{\ell}$. E.g compute $\oint_{\partial S}\vec{A}\cdot d\vec{\ell}$ around a solenoid.

• Aside for later: $\vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$ can be obtained from $L = L_0 + \frac{q}{c}\vec{A}\cdot\vec{v} - q\phi$ (which is nicely relativisitic). Get $\vec{p} = \vec{p}_0 + \frac{q}{c}\vec{A}$. Gives $\frac{d\vec{p}_0}{dt} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$.

• Dirac-Aharonov-Bohm effect: $\psi \sim e^{iS/\hbar} \to$ phase difference $e\Phi/\hbar c$ around a solenoid. Dirac quantization of electric and magnetic monopole charge.

Induced electromagnetic fields

• Define EMF $\mathcal{E} = q^{-1}\oint\vec{F}\cdot d\vec{\ell}$. Faraday's result: $\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt}$. (Minus sign = Lenz's rule, EMF in direction opposing the flux change.)

This is the basis for how power companies make our electricity, and for electric motors: turning wires, in the presence of some magnets.

$\Phi(t)$ can change because of changing $\vec{B}$ and/or changing the loop itself, the result holds in any case. When the loop is fixed, it follows from $\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$.

• Example: moving arm with velocity $\vec{v}$ in presence of constant $\vec{B}_{ext}$. Compute $\mathcal{E} = q^{-1}\oint\vec{F}_{mag}\cdot d\vec{\ell} = -wvB/c = -\frac{1}{c}d\Phi/dt$, where w is the moving arm width. The sign gives the direction relative to $\vec{B}$ and the RHR, e.g. for $\vec{B}$ out of the board, $\mathcal{E}$ is negative because it's clockwise.

• Moving a fixed loop with a velocity $\vec{v}$ through static $\vec{B}$, show $\frac{d\Phi}{dt} = -\oint_C(\vec{v}\times\vec{B})\cdot d\vec{\ell}$, agreeing with the force from $\vec{F}_{mag}$.

• Next time: $\vec{B}$ fields and energy.