1/28/13 Lecture outline

$\star$ Finish chapter 5, start chapter 6.

• Last time, $\mathcal{E} = q^{-1}\oint \vec{F}\cdot d\vec{\ell} = -\frac{1}{c}\frac{d\Phi}{dt}$, and $\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$.

• Recall from before that a magnetic dipole $\vec{m}$ in an external $\vec{B}$ has $U_{dipole} = -\vec{m}\cdot\vec{B}$. This raises the question about magnetic fields doing no work. Indeed, the mechanical work done to bring a current loop $\vec{m}$ into $\vec{B}$ balances the EMF needed to keep the current loop going. Adding the work needed to keep $\vec{B}$ going, get $U_{tot} = +\vec{m}\cdot\vec{B}$. Show it:

Bring current loop in from infinity, in the presence of a big coil, which makes $\vec{B}_{coil}$. Can instead treat the loop as fixed, and bring in the coil. Say $\vec{v}_{loop} = v_{loop}\hat{x}$, Then in time dt, $\Delta\Phi_{loop} = A_{loop}\partial_x B_z v_{loop}dt$, so $\mathcal{E} = -A_{loop}c^{-1}\partial_x B_z v_{loop}$. This $\mathcal{E}$ means that work is required to keep the current I constant, $\Delta W_{emf,loop} = -I\Delta t\mathcal{E} = +m\partial_x B_z v_{loop}\Delta t$, where m is the magnetic moment of the current loop. There is also the force we mentioned before $\vec{F} = \nabla(\vec{m}\cdot\vec{B})$ associated with moving a dipole in an inhomogeneous $\vec{B}$, and some mechanical work is required to push against that, $\Delta W_m = -m\partial_x B_z v_{loop}dt$. So $\Delta W_{e,loop} + \Delta W_m = 0$, as expected, from the fact that magnetic fields do no work on microscopic particles, $\vec{v}\cdot(\vec{v}\times\vec{B}) = 0$. Likewise for the coil, $\Delta W_{e,coil} + \Delta W_m = 0$ (same ΔW_m since the forces are equal and opposite). So $\Delta W_{tot} = \Delta W_{e,loop} + \Delta W_{e,coil} + \Delta W_m = -\Delta W_m$ and $U_{tot} = -U_m = +\vec{m}\cdot\vec{B}$. More generally, writing $d\vec{m} = Id\vec{a}/c$, get

$$U_{tot} = \frac{I_1}{c}\int_{S_1}\vec{B}_2(\vec{r}_1)\cdot d\vec{a}_1 = \frac{I_1}{c}\oint_{\partial S_1}\vec{A}_2(\vec{r}_1)\cdot d\vec{\ell}_1$$

Replace $Id\vec{\ell}\to\vec{J}dV$ and divide by half to avoid double counting

$$U_{tot} = \frac{1}{2c}\int dV\vec{J}\cdot\vec{A} = \frac{1}{8\pi}\int dV(\nabla\times\vec{B})\cdot\vec{A} = \frac{1}{8\pi}\int dV\vec{B}^2.$$

• Another argument: $\Delta J\to\Delta B\to\Delta E$, which oppose ΔJ, so work required,

$$\Delta W = -\Delta t\int dV J\cdot E = \frac{c\Delta t}{4\pi}\int dV\vec{E}\cdot(\nabla\times\vec{B}) = \frac{\Delta t}{4\pi}\int dV\vec{B}\cdot\frac{\partial B}{\partial t} = \Delta\int dV\frac{B^2}{8\pi}.$$

• Inductance: $U_{tot} = \sum_a U_{aa} + \frac{1}{2}\sum_{a\neq b}U_{ab}$, where

$$U_{aa} = \frac{1}{2c}\int dV\vec{J}_a\cdot\vec{A}_a = \frac{1}{2c^2}\int dV\int dV'\frac{\vec{J}_a(\vec{r})\cdot\vec{J}_a(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

and likewise for U_{ab}, but without the $\frac{1}{2}$. Now $U_{tot} = \frac{1}{2}\sum_{ab}L_{ab}I_aI_b$, with $L_{ab}\approx\frac{1}{c^2}\oint\oint d\vec{\ell}\cdot d\vec{\ell}'/|\vec{r}-\vec{r}'|$. Note $L_{ab} = L_{ba}$. Also flux through loop a from current in loop b is $\Phi_a = cL_{ab}I_b$, so EMF is $\mathcal{E}_a = -L_{ab}\dot{I}_b$. Units: $U = \frac{1}{2}LI^2$ in either set, so $L_{SI} = \frac{\mu_0 c^2}{4\pi}L_{Gau} = \frac{1}{4\pi\epsilon_0}L_{Gau}$, and L_{SI} is in Henry units, with $1 Henry = \frac{1}{9}\times 10^{11}$ Gaussian units.

E.g. solenoid of length h, and n turns/length and radius R has $B_{in} \approx 4\pi nI/c$, so $U = \int B^2/8\pi = \frac{1}{2}LI^2$, with $L = h(2\pi nR/c)^2$.

E.g. let R_1 be the radius of a small loop and R_2 that of a big loop, which are concentric and in a plane. Use $\Phi_a = cL_{ab}I_b$: flux of the big loop's I through the small one is $\pi R_1^2(2\pi I_2)/cR_2$ so the mutual inductance is $L_{12} \approx 2\pi^2 R_1^2/c^2 R_2$ (for $R_1 \ll R_2$). Note $L_{12} = L_{21}$, even though it isn't obvious (because of the limit).