1/30/13 Lecture outline

• Conservation laws and symmetries. Charge conservation, in differential and integral form.

• Changing $\vec{E}$ as a source of $\vec{B}$: $\oint \vec{B}\cdot d\vec{\ell} = \frac{4\pi}{c} I_{encl} + \frac{1}{c}\frac{d\Phi_{elec}}{dt}$, satisfies charge conservation. Example: consider parallel plate capacitor, charging up, showing the RHS gives the same answer whether the surface has I_{encl} or has $\dot{\Phi}_{elec} \neq 0$. Illustrates

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}.$$

This has charge conservation built in: taking the divergence gives $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$.

• Now the potentials are $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$. Gauge invariance: $\vec{A} \to \vec{A}' = \vec{A} + \nabla f$ and $\phi \to \phi' = \phi - \frac{1}{c}\frac{\partial f}{\partial t}$ preserves $\vec{E}$ and $\vec{B}$. Interesting: all of physics is invariant under such changes; non-trivial in quantum mechanics. Related to charge conservation.

The equations to solve are now

$$-\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \vec{A} = 4\pi\rho$$

$$-\nabla^2\vec{A} + \nabla(\nabla \cdot \vec{A}) + \frac{1}{c}\frac{\partial}{\partial t}\nabla\phi + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J}.$$

Coulomb gauge: $\nabla \cdot \vec{A} = 0$; gives $\phi(\vec{x},t) = \int dV' \rho(\vec{x}',t)/|\vec{x}-\vec{x}'|$, instantaneous (bad with relativity). Lorentz gauge: $\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$ (preserved by Lorentz transformations, so nice in relativity).

• Examples showing need for $\vec{P}_{field}$ and $\vec{L}_{field}$. Example of solenoid with charge Q at radius R, show $L = \frac{1}{2}I\dot{\theta}^2 + \frac{q}{c}\dot{\theta}RA_\theta$, with $2\pi RA_\theta = \Phi_{mag}$, the magnetic flux. So $p_\theta = I\dot{\theta} + q\Phi_{mag}/2\pi c$ is a constant in t. If the wire in the current stops, $\dot{\theta}$ compensates for reduced Φ_{mag}, the thing starts spinning. Also see it from the angular momentum impulse from the torque from the EMF from $\mathcal{E} = -\frac{1}{c}\dot{\Phi}_{mag}$.

• Energy flow. The field energy density is $\mathcal{U}_{field} = (\vec{E}^2 + \vec{B})/8\pi$. Note that

$$\frac{\partial \mathcal{U}_{field}}{\partial t} = \vec{E}\cdot(\frac{c}{4\pi}\nabla \times \vec{B} - \vec{J}) - \vec{B}\cdot\frac{c}{4\pi}\nabla \times \vec{E}.$$

$$\frac{\partial \mathcal{U}_{field}}{\partial t} = -\vec{J}\cdot\vec{E} - c\nabla \cdot \frac{\vec{E}\times\vec{B}}{4\pi}.$$

Note $\vec{J}\cdot\vec{E} = \frac{d}{dt}\mathcal{E}_{kin}$ (since $q\vec{v}\cdot\vec{E} = \vec{v}\cdot\frac{d\vec{p}}{dt} = dE_{kin}/dt$), so energy conservation is

$$\frac{d}{dt}\left[\int_V dV(\mathcal{U}_{field} + \mathcal{E}_{kin})\right] + \int_{\partial V} \vec{S}\cdot d\vec{a} = 0.$$

$\vec{S} = c\vec{E}\times\vec{B}/4\pi$ is the energy flux density. Also $\int dV\vec{J}\cdot\vec{E}$ is the received mechanical power.

- For an electron, both q and $\vec{m} \neq 0$, so $\vec{S} = qc\vec{m}\times\vec{r}/4\pi r^6$. Note $\vec{S}\cdot\hat{r} = 0$.
- Examples. Charging capacitor: show $U \approx \frac{1}{8\pi}\int dV\vec{E}^2 = Q^2/2C$, and $\int_{\partial V}\vec{S}\cdot d\vec{a} = -\frac{c}{4\pi}\Delta\phi\oint\vec{B}\cdot d\vec{\ell} = -\Delta\phi\dot{Q}$. Next example: starting up a solenoid.