3/2/15 -3/4/15 week, Ken Intriligator's Phys 4D Lecture outline

• The principle of special relativity is that all of physics must transform between different inertial frames such that all are equally valid. No experiment can tell Alice or Bob which one is moving, as long as both are in inertial frames (so v_{rel} = constant).

• As we discussed, if $a^\mu = (a^0, \vec{a})$ and $b^\mu = (b^0, \vec{b})$ are any 4-vectors, they transform with the usual Lorentz transformation between the lab and rocket frames, and $a \cdot b \equiv a^0 b^0 - \vec{a} \cdot \vec{b}$ is a Lorentz invariant quantity, called a 4-scalar. We've so far met two examples of 4-vectors: x^μ (or dx^μ) and k^μ.

• Some other examples of 4-scalars, besides Δs^2: mass m, electric charge q. All inertial observers can agree on the values of these quantities. Also proper time, $d\tau \equiv \sqrt{ds^2/c^2}$.

• Next example: 4-velocity $u^\mu = dx^\mu/d\tau = (\gamma, \gamma\vec{v})$. Note $\frac{d}{d\tau} = \gamma \frac{d}{dt}$. The addition of velocities formula becomes the usual Lorentz transformation for 4-velocity. In a particle's own rest frame, $u^\mu = (1, \vec{0})$. Note that $u \cdot u = 1$, in any frame of reference. u^μ can be physically interpreted as the unit tangent vector to a particle's world-line. All $\vec{v}$'s in physics should be replaced with u^μ.

• In particular, $\vec{p} = m\vec{v}$ should be replaced with $p^\mu = mu^\mu$. Argue $p^\mu = (E/c, \vec{p})$ from their relation to x^μ. So $E = \gamma mc^2$ and $\vec{p} = \gamma m\vec{v}$. For non-relativistic case, $E \approx mc^2 + \frac{1}{2}m\vec{v}^2$, so rest-mass energy and kinetic energy. For massless particles, e.g. the photon, $p^\mu \neq 0$ (and u^μ is ill-defined since $d\tau = 0$).

• Aside: next quarter, you'll learn $E = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$ so we can write $p^\mu = \hbar k^\mu$ as a 4-vector equation, so it fits with relativity. The quantum wavefunction for a free particle looks like a traveling wave $\psi \sim e^{i\vec{k}\cdot\vec{x} - \omega t} = e^{ip\cdot x/\hbar}$. (History aside: Einstein '05 for light, De Broglie 1925 for electrons and other matter. Einstein's $E = \hbar\omega$ was widely thought to be wacky and wrong, until it was experimentally verified around 1916 by Millkan, who set out to disprove it; Einstein's 1921 Nobel prize was officially that, in the photoelectric effect.)

• $p \cdot p = (E/c)^2 - \vec{p}^2$ is the same in all inertial frames. For a massive object, can go to its rest frame to find $p \cdot p = (mc)^2$. For a massless particle (e.g. the photon), $p \cdot p = 0$. Can never go to a massless particle's rest frame, always moving at $v = c$. Whether massive or massless, we have $E = \sqrt{c^2\vec{p}^2 + (mc^2)^2}$. For $p \ll mc$, we can Taylor expand this using again $(1+x)^n \approx 1 + nx + n(n-1)x^2/2 + \ldots$ to find again $E \approx mc^2 + (p^2/2m) - (p^4/8m^3c^2) + \ldots$.

• Energy and momentum vector as tangent to worldline, and energy-momentum hyperbola for different frames.

• Note that $\vec{p}/E = \vec{v}$. This is true even for massless particles, e.g. photons.

• $m_P c^2 \approx 938 MeV \approx 1 GeV$. At the LHC (after upgrade), the protons will have $E = 6.5 TeV$, so $\gamma \approx 6.5 \times 10^3$. Cosmic rays have even bigger energies. Suppose a cosmic ray proton has $E = 10^{11} GeV$ (as is observed). So $\gamma \approx 10^{11}$ (more useful than quoting v/c, which is extremely close to 1). In this limit, $p \approx \gamma mc$. The galaxy is $\approx 10^5$ light years, so such protons cross the galaxy in about 10^{-6}years ≈ 31.5 seconds according to their own proper time.

• Proton made up of three quarks, plus glue. The rest masses of the quarks give only about 1% of the proton's mass. The rest is the energy of the moving quarks and glue.

• Replace $\vec{F} = m\vec{a}$ with $f^\mu = m\frac{dp^\mu}{d\tau}$, where f^0 is related to power. For $f^\mu_{ext} = 0$, get p^μ constant. Conservation of overall energy and momentum of a closed system (e.g. the universe): $\sum_{\text{allparticles}} p^\mu = \text{constant}^\mu$ (i.e. all 4 components are constant).

• Brief aside: acceleration 4-vector $a^\mu = d^2x^\mu/d\tau^2$, vs $\vec{a}$, and Lorentz transformations of them.

• Example: show that $e^+ + e^- \to \gamma$ is not possible. $e^+ + e^- \to \gamma + \gamma$ is, converting the $E = mc^2$ of matter into energetic photons. Will be studied more in the HW. (History aside: the positron, = anti-electron, was predicted by Dirac, from his relativistic quantum theory in 1928. Originally many thought it was a defect of the theory, then it was experimentally found in 1932 by Carl Anderson. The origin of the asymmetry between matter and antimatter in the universe is an interesting question.)

• Compton effect: $\gamma + e^- \to \gamma + e^-$. Take e^- to be at rest in the lab frame, and the photon to have initial energy E_0, traveling along the x-axis. The outgoing photon makes an angle θ with respect to the x axis and has energy E. Find E: by squaring $p_1^\mu + p_2^\mu - p_3^\mu = p_4^\mu$, where $p_{1,3}$ refer to the photon and $p_{2,4}$ to the electron: $E = E_0(1 + (E_0/m_0c^2)(1 - \cos\theta))^{-1}$

• Example: A particle of mass $5m$, at rest, decays into a particle of mass m and one of mass $2m$ find their energies and momenta: $E_1 = 11m/5$, $E_2 = 14m/5$, $p_{1,2} = \mp\sqrt{96}(mc)/5$.

• Example: suppose a particle of mass m, and energy E, hits a particle of mass M that was initially at rest. Suppose that they stick together and make an new particle, of mass M' and energy E'. Find M' and E': $p_1^\mu + p_2^\mu = p_3^\mu$, where $p_1^2 = (mc)^2$, $p_2^2 = (Mc)^2$, and $p_3^2 = (M'c)^2$. Square both sides of the 4-vector equation: $(mc)^2 + (Mc)^2 + 2EM = (M'c)^2$.

• Example: a particle of mass $3m$ and energy $5m$ has an elastic collision with a particle of mass m that is at rest. Find their energy and momentum in the CM frame. Lorentz transform via $\frac{3}{\sqrt{5}}\begin{pmatrix} 1 & -2/3 \\ -2/3 & 1 \end{pmatrix}$ to get $p_1' = \frac{1}{\sqrt{5}}\begin{pmatrix} 7 \\ 2 \end{pmatrix}$ and $p_2' = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. After the collision, just flip the sign of the spatial momentum. Can then Lorentz transform back to the lab frame.