2/23/16 Lecture 13 outline

• Last time we wrote down the LSZ formula. There was some interest in seeing more details, so let's briefly sketch the idea.

Let $|k\rangle$ be the physical one-particle momentum plane wave state of the full interacting theory, normalized to $\langle k'|k\rangle = (2\pi)^3 2\omega_k \delta^{(3)}(\vec{k}' - \vec{k})$, and $\phi(x)$ the Heisenberg picture field. As discussed last time, the FT of $\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle \sim iZ/(p^2 - m^2 + i\epsilon)$ near $p^2 = m^2$, so

$$\langle k|\phi(x)|\Omega\rangle = \langle k|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\Omega\rangle = e^{ik\cdot x}\langle k|\phi(0)|\Omega\rangle \equiv e^{ik\cdot x} Z_\phi^{1/2}.$$

We scatter wave packets, with some profile $F(\vec{k})$, with F.T. $f(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(\vec{k}) e^{-ik\cdot x}$, where we define $k_0 = \sqrt{\vec{k}^2 + \mu^2}$, so $f(x)$ solves the KG equation. Now define

$$\phi^f(t) = iZ_\phi^{-1/2} \int d^3\vec{x}(\phi(\vec{x},t)\partial_0 f(\vec{x},t) - f(\vec{x},t)\partial_0\phi(\vec{x},t)).$$

This depends only on t, and we'll be interested in it at $t \to \pm\infty$, where it makes asymptotic **single-particle** in and out states: $\langle k|\phi^f(t)|\Omega\rangle = F(\vec{k})$ (the ∂_0's in $\phi^f(t)$ give a needed $2\omega_k$ to cancel that in $d^3k/(2\pi)^3 2\omega_k$), and $\langle n|\phi^f(t)|\Omega\rangle = \frac{\omega_{p_n} + p_n^0}{2\omega_{p_n}} F(\vec{p}_n) e^{-i(\omega_{p_n} - p_n^0)t}\langle n|\phi(0)|\Omega\rangle$, where $\omega_{p_n} \equiv \sqrt{\vec{p}_n^2 + \mu^2}$, which has $\omega_{p_n} < p_n^0$ for any multiparticle state. So for **any** state ψ, $\lim_{t\to\pm\infty}\langle\psi|\phi^f(t)|\Omega\rangle = \langle\psi|f\rangle + 0$, where $|f\rangle \equiv \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} F(\vec{k})|\vec{k}\rangle$, and the multiparticle states contributions sum to zero using the Riemann-Lebesgue lemma. Moreover, you can easily verify that (taking $f(|x| \to \infty) \to 0$)

$$iZ_\phi^{-1/2} \int d^4x f(x)(\partial^2 + \mu^2)\phi(x) = \int dt \partial_0 \phi^f(t) = (\lim_{t\to-\infty} - \lim_{t\to\infty})\phi^f(t).$$

This will be just what we wanted, to get our incoming and outgoing scattering states.

Make separated in states: $|f_n\rangle = \prod \phi^{f_n}(t_n)|\Omega\rangle$, and out states $\langle f_m| = \langle\Omega|\prod(\phi^{f_m})^\dagger(t_m)$, with $t_n \to -\infty$ and $t_m \to +\infty$. With some work, it can be shown that the $|_{-\infty}^{\infty}$ differences work out right so that

$$\langle f_m|S - 1|f_n\rangle = Z_\phi^{-(n+m)/2} \int \prod_n d^4x_n f_n(x_n) \prod_m d^4x_m f_m(x_m)^* \prod_r i(\partial_r^2 + m_r^2) G(x_n, x_m).$$

Take $f_i(x) \to e^{-ik_i x_i}$ at the end. Thus get that the S-matrix element for m incoming particles and n outgoing ones is given by

$$\langle \mathbf{p_1} \ldots \mathbf{p_n}|S|\mathbf{k_1} \ldots \mathbf{k_m}\rangle = Z_\phi^{-(n+m)/2} \lim_{o.s} \prod_{i=1}^n (p_i^2 - m_i^2) \prod_{j=1}^m (k_j^2 - m_j^2) \tilde{G}^{n+m}(-p_i, k_i).$$

Again, $\tilde{G}^{n+m}$ is the full $n+m$ point Green's function, including disconnected diagrams etc. The limit is where we take the external particles on shell. In this limit, the $p_i^2 - m_i^2$ and $k_j^2 - m_j^2$ prefactors all go to zero. These zeros kill everything on the RHS except for the connected contributions to $\tilde{G}$. Accounting for the fact that we amputate the external propagators, which go like $iZ_i(p_i^2 - m_i^2)^{-1}$, the above becomes

$$\langle \mathbf{p_1} \dots \mathbf{p_n} | S | \mathbf{k_1} \dots \mathbf{k_m} \rangle = Z^{(n+m)/2} \tilde{G}^{n+m}_{amp,conn,B}(-p_i, k_i) = \tilde{G}^{n+m}_{amp,conn,R}(-p_i, k_j)$$

Good: the physical S-matrix elements are computed from the renormalized Greens functions, which we take to be finite in our renormalization procedure.

• Write

$$-i\widetilde{\Delta}(p^2) = \frac{i}{p^2 - m^2 - \Pi'(p^2) + i\epsilon} = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{\sim 4m^2}^{\infty} \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon}.$$

So, using $\frac{1}{x \pm i\epsilon} = P(1/x) \mp i\pi\delta(x)$, argue that $\pi\rho(s) = 2Im\widetilde{\Delta}(s)$ for $s \geq 4m^2$. (The minus sign in the definition of $\widetilde{\Delta}$ above is related to the special definition of $\widetilde{\Gamma}^{(n)}$ for $n=2$ and $\widetilde{\Delta} \sim 1/\widetilde{\Gamma}^{(2)}$.)

Analyticity properties. E.g. $2 \to 2$ scattering. $\mathcal{M}(s) = \mathcal{M}(s^*)^*$. The real part $Re\mathcal{M}$ is continuous across the real axis, whereas the Im part picks up a minus sign. So the discontinunity $Disc\mathcal{M}(s) = 2iIm\mathcal{M}(s + i\epsilon)$. E.g. $\frac{1}{x \pm i\epsilon} = P(1/x) \mp i\pi\delta(x)$ shows that the discontinunity of $\frac{1}{p^2 - m^2 + i\epsilon}$ is $-2\pi i\delta(p^2 - m^2)$.

• Optical theorem. The S-matrix $S = U(t_f = \infty, t_i = -\infty)$ is unitary, $S^\dagger S = 1$. Write $S = 1 + iT$, then get $2Im(T) \equiv -i(T - T^\dagger) = T^\dagger T$. Thus

$$-i(2\pi)^4\delta^4(p_f - p_i)(\mathcal{M}_{fi} - \mathcal{M}^*_{if}) = \sum_m \prod_j \int \frac{d^3\vec{k}_j}{(2\pi)^3 2E_j} \mathcal{M}_{fm}\mathcal{M}^*_{im}(2\pi)^4\delta^4(p_f - p_m)(2\pi)^4\delta^4(p_f - p_i).$$

Take $f = i$, get

$$2Im\mathcal{M}_{ii} = \sum_m \int d\Pi_m |\mathcal{M}_{im}|^2,$$

where $d\Pi_m$ is the density of states for the process $i \to m$. This is the optical theorem. It relates the imaginary part of the forward scattering amplitude to the total cross section, e.g.

$$\mathrm{Im}\mathcal{M}(\mathrm{k}_1, \mathrm{k}_2 \to \mathrm{k}_1, \mathrm{k}_2) = 2\mathrm{E}_{\mathrm{cm}}\mathrm{p}_{\mathrm{cm}}\sigma_{\mathrm{tot}}(\mathrm{k}_1, \mathrm{k}_2 \to \text{anything}).$$

Recall that the imaginary part of amplitudes is discontinuous across the cut starting at $s = 4m^2$. So we can there relate

$$Disc\mathcal{M}(s) = 2iIm\mathcal{M}(s) \sim \int d\Pi \, |\mathcal{M}_{cih}|^2 \sim \sigma_{tot}$$

where cih means cut in half.

Consider e.g. the 1-loop contribution to the 4-point amplitude in $\lambda\phi^4$, in the s channel

$$\mathcal{M}^{(1)} = \tfrac{1}{2}\lambda^2 \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(\frac{1}{2}p+k)^2 - m^2 + i\epsilon} \frac{1}{(\frac{1}{2}p-k)^2 - m^2 + i\epsilon},$$

where $p = p_1 + p_2$. Recall that we evaluated this as (with $s = p^2$)

$$\frac{\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\frac{4\pi\mu^2}{m^2} + A(s),\right)$$

where

$$A(s) = 2 - \sqrt{1 - 4m^2/s}\log\left(\frac{\sqrt{1-4m^2/s}+1}{\sqrt{1-4m^2/s}-1}\right).$$

The $1/\epsilon$ term (together with some constants, depending on our scheme) is cancelled by a counterterm diagram. The function $A(s)$ remains. The threshold is where $s = 4m^2$. Below threshold, the amplitude is purely real. Above threshold, there is a discontinuous imaginary part, with

$$Disc\mathcal{M}(s) = 2iIm\mathcal{M}(s) \sim \int d\Pi \, |\mathcal{M}_{cih}|^2 \sim \sigma_{tot}$$

where cih means cut in half. The tree-level scattering amplitude is thus related to the imaginary part of the one-loop amplitude.

• For unstable particles, we can again write the full propagator as $i(p^2 - m^2 - \Pi'(p^2))^{-1}$, and the decay width again shows up via an analog of the optical theorem for 1-particle to 1-particle scattering. This gives the decay width, which appears in the Breit-Wigner formula $\sigma \sim |p^2 - m^2 + i\Gamma|^{-2}$, as $\Gamma = -m^{-1}ZIm\Pi'(p^2) = \frac{1}{2m}\sum_f \int d\Pi_f |\mathcal{M}(p \to f)|^2$.

•Let's consider more generally

$$\tilde{\Gamma}_B^{(n)}(p_1, \ldots p_n; \lambda_B, m_B, \epsilon) = Z_\phi^{-n/2}\tilde{\Gamma}_R^{(n)}(p_1, \ldots p_n; \lambda_R, m_R, \mu, \epsilon).$$

For fixed physics, the LHS is some fixed quantity. The RHS depends on the renormalization point μ and the scheme. The LHS does not! This leads to what is known as the renormalization group equations, which state how the renormalized quantities must vary with μ.

Take $d/d\ln\mu$ of both sides, and use $d\Gamma_B/d\mu = 0$. This gives

$$\left(\frac{\partial}{\partial\ln\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} + \gamma_m m_R\frac{\partial}{\partial\ln m_R} - n\gamma\right)\tilde{\Gamma}_R^{(n)}(p_1,\ldots p_n;\lambda_R,m_R,\mu) = 0$$

Here

$$\beta(\lambda) \equiv \frac{d}{d\ln\mu}\lambda_R$$

$$\gamma = \tfrac{1}{2}\frac{d}{d\ln\mu}\ln Z_\phi$$

$$\gamma_m = \frac{d\ln m_R}{d\ln\mu}.$$

This is the RG equation. Various variants, depending on subtraction procedure (scheme). For mass dependent scheme, this gives the original Gell-Mann Low equations, where β and γ depend on the physical mass. The Callan-Symanzik equation replaces $\partial/\partial\ln\mu$ with $\partial/\partial\ln m$, giving the change as the physical mass is varied. It's often better to use a mass-independent scheme, like MS (or $\overline{MS}$, where we had introduced the scale M in replacing, via appropriate counterterms, $(\frac{2}{\epsilon} - \gamma + \log(4\pi/m^2) \to \log(M^2/m^2))$, where m appears as just another coupling constant. In any case, the RG equation can be integrated, to relate the renormalized Greens functions at different scales μ and μ'.