2/1/16 Lecture 8 outline

• Last time:

$$I_n(a) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+a)^n}$$

with n integer and $Im(a) > 0$ and k in Minkowski space. See

$$I_n = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{da^{n-1}} I_1(a), \qquad I_1 = \frac{-i}{16\pi^2} \int_0^{\Lambda^2} du \frac{u-a+a}{u-a}$$

where we used the solid angle $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$, which is $2\pi^2$ for $D=4$. Get

$$I_n(a) = i\left(16\pi^2(n-1)(n-2)a^{n-2}\right)^{-1} \qquad \text{for} \quad n \geq 3.$$

Special cases

$$I_1 = \frac{i}{16\pi^2} a \ln(-a) + \ldots,$$

$$I_2 = \frac{-i}{16\pi^2} \ln(-a) + \ldots,$$

where $\ldots$ are terms involving the regulator.

• Let's illustrate another, extremely popular, choice of regulator: dimensional regularization. Suppose that we had D instead of 4 dimensions. Compute by analytic continuation in D. Then take $D = 4 - \epsilon$, and take $\epsilon \to 0$. By going slightly below 4 dimensions, we improve the UV behavior (make the theory weaker in the UV, though stronger in the IR). In particular, using the notation above,

$$I \equiv iI_1(-m^2) \equiv \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{k_E^2 + m^2} = \frac{\Omega_{D-1}}{(2\pi)^D} \int_0^\infty u^{D-1} du \frac{1}{u^2+m^2}.$$

Again, $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of a unit sphere S^{D-1}. Let $u^2 = m^2 y$

$$I = \frac{m^{D-2}}{2^D \pi^{D/2} \Gamma(D/2)} \int_0^\infty \frac{y^{(D-2)/2} dy}{y+1}.$$

Now use $(y+1)^{-1} = \int_0^\infty dt e^{-t(y+1)}$ and $\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1}$ to get

$$I = \frac{m^{D-2}}{(4\pi)^{D/2}} \Gamma(1 - \tfrac{1}{2}D).$$

This blows up for $D = 4$, because $\Gamma(1 - \frac{1}{2}D)$ has a pole there. Recall $\Gamma(z)$ has a simple pole at $z = 0$, and also at all negative integer values of z.

Recall that near $x = 0$,

$$\lim_{x\to 0}\Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(x),$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. For $x = -n$, we can write a similar expression, which also follows from the above and $\Gamma(z+1) = z\Gamma(z)$. This gives

$$\lim_{x\to -n}\Gamma(x) = \frac{(-1)^n}{n!}(\frac{1}{x+n} - \gamma + 1 + \ldots + \frac{1}{n} + \mathcal{O}(x+n).).$$

E.g. use $\Gamma(2 - D/2) = (1 - D/2)\Gamma(1 - D/2)$. Let $D = 4 - \epsilon$, then (dropping $\mathcal{O}(\epsilon)$,

$$\frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}}\Delta^{D/2-2} \to \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log\frac{\Delta}{4\pi} - \gamma\right).$$

We can apply this to evaluate $\Pi^{(1)}(p^2) = \frac{1}{2}\lambda I$. One last thing: replace $\lambda_{old} = \lambda_{new}\mu^{4-D}$, where λ_{new} is dimensionless. Expanding around $D = 4$, we get

$$\Pi'(p^2)^{(1)} = -\frac{\lambda m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \log\frac{m^2}{4\pi\mu^2} + 1 - \gamma\right).$$

The scale μ introduced above, which we'll see is immaterial at the end of the day, nicely makes the units work inside the log. Summarizing, at one-loop there is a $1/\epsilon$ pole, which we'll deal with soon, and a finite piece.

• More useful integrals:

$$\int\frac{d^Dk_E}{(2\pi)^D}\frac{1}{(k_E^2 + \Delta)^n} = \frac{1}{4\pi)^{D/2}}\frac{\Gamma(n - \frac{1}{2}D)}{\Gamma(n)}\Delta^{D/2-n}.$$

$$\int\frac{d^Dk_E}{(2\pi)^D}\frac{k_E^2}{(k_E^2 + \Delta)^n} = \frac{1}{4\pi)^{D/2}}\frac{D}{2}\frac{\Gamma(n - \frac{1}{2}D - 1)}{\Gamma(n)}\Delta^{1+D/2-n}.$$

• Now consider $\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4)$. There are three 1-loop diagrams, in the s, t, u channels. Recall $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_1 + p_4)^2$, $s + t + u = 4m^2$. Get

$$\tilde{\Gamma}^{(4)} = -\lambda\hbar^{-1} + (-i\lambda)^2(F(s) + F(t) + F(u)) + O(\hbar),$$

where

$$F(p^2) = \tfrac{1}{2}i\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - m^2}\frac{1}{(k+p)^2 - m^2}.$$

The $\frac{1}{2}$ is a symmetry factor. Evaluate using

$$\frac{1}{AB} = \int_0^1 dx\frac{1}{(xA + (1-x)B)^2}.$$

Aside: more generally, have

$$\prod_{j=1}^n A_j^{-\alpha_j} = \frac{\Gamma(\sum_j \alpha_j)}{\prod_j\Gamma(\alpha_j)}\int_0^1 dx_1\ldots\int_0^1 dx_n\delta(1 - \sum_j x_j)\frac{\prod_k x^{\alpha_k - 1}}{(\sum_i x_iA_i)^{\sum\alpha_j}}.$$