2/2/16 and 2/4/16 Lectures 9 and 10 outline

- Last time, lots of integrals:

$$\int \frac{d^D k_E}{(2\pi)^D} \frac{1}{(k_E^2+\Delta)^n} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(n-\frac{1}{2}D)}{\Gamma(n)} \Delta^{D/2-n}.$$

$$\int \frac{d^D k_E}{(2\pi)^D} \frac{k_E^2}{(k_E^2+\Delta)^n} = \frac{1}{4\pi)^{D/2}} \frac{D}{2} \frac{\Gamma(n-\frac{1}{2}D-1)}{\Gamma(n)} \Delta^{1+D/2-n}.$$

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA+(1-x)B)^2}.$$

Aside: more generally, have

$$\prod_{j=1}^n A_j^{-\alpha_j} = \frac{\Gamma(\sum_j \alpha_j)}{\prod_j \Gamma(\alpha_j)} \int_0^1 dx_1 \ldots \int_0^1 dx_n \delta(1-\sum_j x_j) \frac{\prod_k x^{\alpha_k -1}}{(\sum_i x_i A_i)^{\sum \alpha_j}}.$$

Also,

$$\lim_{D=4-\epsilon} \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}} \Delta^{D/2-2} \to \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \log \frac{\Delta}{4\pi} - \gamma \right).$$

As examples, we applied these to $\lambda\phi^4$ theory and obtained

$$\Pi'(p^2)^{(1)} = -\frac{\lambda m^2}{32\pi^2} \left(\frac{2}{\epsilon} - \log \frac{m^2}{4\pi\mu^2} + 1 - \gamma \right).$$

$$\tilde{\Gamma}^{(4)} = -\lambda\hbar^{-1} + (-i\lambda)^2 (F(s)+F(t)+F(u)) + O(\hbar),$$

where

$$F(p^2) = \tfrac{1}{2} i \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2-m^2} \frac{1}{(k+p)^2-m^2}.$$

Recall that for $n > 2$

$$\text{1PI diagram} \equiv i\tilde{\Gamma}^{(n)}(p_1, \ldots p_n),$$

while for $n=2$ the 1PI diagram is $-i\Pi'(p)$ and the full propagator is obtained by the geometric series sum to be

$$D(p) = \frac{i}{p^2-m^2-\Pi'(p^2)+i\epsilon} \equiv \frac{i}{\tilde{\Gamma}^{(2)}}.$$

- OK, continue where we left off:

$$F(p_E^2) = -\tfrac{1}{2} \int \frac{d^4 k_E}{(2\pi)^4} \int_0^1 dx \frac{1}{(xk_E^2+m^2+(1-x)(k_E+p_E)^2)^2}.$$

The quantity in the denominator is $k_E^2 + (1-x)2k_E \cdot p_E + (1-x)p_E^2 + m^2 = (k_E + (1-x)p_E)^2 + p_E^2(1-x)x + m^2$, so

$$F(s_E) = -\tfrac{1}{2}\int \frac{d^4 k_E}{(2\pi)^4}\int_0^1 dx \frac{1}{(k_E^2 + m^2 + x(1-x)s_E)^2}.$$

Where $s_E = p_E^2 = -s$. Evaluate the k integral using the dimreg integrals above. Expanding around $D = 4 - \epsilon$, this gives

$$F(s_E) = -\frac{1}{32\pi^2}\int_0^1 dx\left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log(m^2 + x(1-x)s_E)\right).$$

So the one-loop contribution to $\widetilde{\Gamma}^{(4)}$ is

$$\frac{\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\frac{4\pi\mu^2}{m^2} - \int_0^1 dx \log(1 + x(1-x)\frac{s_E}{m^2})\right) + (s \to t) + (s \to u).$$

The integral is evaluated using

$$\int_0^1 dx \log(1 + \frac{4}{a}x(1-x)) = -2 + \sqrt{1+a}\log\left(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\right) \qquad a > 0.$$

So finally,

$$\widetilde{\Gamma}^{(4)} = -\lambda\hbar^{-1} + \frac{\lambda^2}{32\pi^2}\left(3\frac{2}{\epsilon} - 3\gamma + 3\log\frac{4\pi\mu^2}{m^2} + A_1(s) + A_1(t) + A_1(u)\right) + \mathcal{O}(\lambda^3\hbar)$$

with again a $1/\epsilon$ pole and a finite term at one loop, with

$$A_1(s) = 2 - \sqrt{1 - \frac{4m^2}{s}}\log\left(\frac{\sqrt{1-\frac{4m^2}{s}}+1}{\sqrt{1-\frac{4m^2}{s}}-1}\right).$$

The finite terms have interesting behavior at $s, t, u = 4m^2$, which as we'll discuss is related to intermediate channel particles going on-shell.

• Renormalization. The input to the functional integral is the "bare" lagrangian. It is not physically observable, because we observe quantities like mass, charge, etc. with all the quantum corrections included. Write the largrangian for the bare fields as:

$$\mathcal{L}_B = \tfrac{1}{2}\partial_\mu\phi_B\partial^\mu\phi_B - \tfrac{1}{2}m_B^2\phi_B^2 - \lambda_B\frac{1}{4!}\phi_B^4.$$

The bare field is related to the physical one by $\phi_B \equiv Z_\phi^{1/2}\phi$. We can view this as

$$\mathcal{L}_B = \mathcal{L}_{phys} + \mathcal{L}_{c.t.}$$

where

$$\mathcal{L}_{phys} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \lambda\mu^\epsilon\frac{1}{4!}\phi^4$$

involves the physical field, mass, coupling constant. What's left are the counterterms:

$$\mathcal{L}_{c.t.} = \tfrac{1}{2}(Z_\phi - 1)\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}(m_B^2 Z_\phi - m^2)\phi^2 - (\lambda_B Z_\phi^2 - \lambda\mu^\epsilon)\frac{1}{4!}\phi^4.$$

Define $\delta_Z \equiv Z_\phi - 1$, $\delta_m = m_B^2 Z_\phi - m^2$, $\delta_\lambda\mu^\epsilon = \lambda_B Z_\phi^2 - \lambda\mu^\epsilon$. There are extra diagram contributions for these corrections.

There is a line (like the propagator) with an insertion of the counterterm, which gives a factor of $i(p^2\delta_Z - \delta_m)$. There is a new vertex with a factor of $-i\delta_\lambda$. These new diagrams count as having one loop factor (one factor of $\hbar$).

• Among other things, these corrections cancel the divergences. E.g. δm *adds* to Π', so pick the additive contribution to cancel the divergence in Π'; likewise, $\delta\lambda$ adds to effective λ obtained from $\widetilde{\Gamma}^{(4)}$, so

$$\delta_m = \frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon} + \text{finite} + \mathcal{O}(\lambda^3).$$

$$\delta_\lambda = 3\frac{\lambda^2}{16\pi^2}\frac{1}{\epsilon} + \text{finite} + \mathcal{O}(\lambda^4).$$

To one loop, $\delta_Z = 0+$(finite), because $\Pi'(p^2)$ is independent of p^2.

Non-trivial fact: we can cancel every divergence in $\lambda\phi^4$, just by using δZ, δm^2, and $\delta\lambda$. Contrast this with $\lambda_6\phi^6$, where more and more counterterms are required, e.g. the 1-loop contribution to $\widetilde{\Gamma}^{(8)}$ requires a $\delta\lambda_8\phi^8$ counterterm, and it's never ending. Renormalizable vs non-renormalizable theories.

• Renormalizability: all divergences cancelled by counter terms of the same form as original $\mathcal{L}$. This would not be the case for e.g. $\lambda\phi^6$. Even for $\lambda\phi^4$, it is quite non-trivial. For example, in doing 2 loops, there could have been some term from one loop diagrams, with counter terms, leading to $\frac{1}{\epsilon}\ln p^2$, which could not be cancelled by a counterterm in our lagrangian. Sometimes individual diagrams indeed behave like that. But the coefficients of all such terms sum to zero.

• What to do about the finite parts is a choice that we can make, called our renormalization prescription. We have to define what we're calling the physical mass and coupling. The physics will be independent of our particular choice, and different choices have different calculational advantages or disadvantages. We'll discuss three choices: (i) on shell; (ii) minimal subtraction (MS); (iii) $\overline{MS}$.

• On shell renormalization scheme. Here, we define what we mean by the mass to be the pole of the full propagator (sum of all connected diagrams), $D(p) = i/\tilde{\Gamma}^{(2)}$, and to define the physical field so that the residue of the pole is i. This means

$$\Pi'(m^2) = 0, \qquad \frac{d\Pi'}{dp^2}|_{p^2=m^2} = 0, \qquad \tilde{\Gamma}^{(4)}|_{s=4m^2} = -\lambda$$

where the last condition is our definition of physical λ. With this choice, we have

$$\delta_m = +\frac{\lambda m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \log\frac{m^2}{4\pi\mu^2} + 1 - \gamma\right)$$

to this order, and so, to this order they combine to give

$$\Pi'(p^2) = 0.$$

We also have $\delta_Z^{(1)} = 0$ and $\delta_\lambda^{(1)}$ is such that now

$$\tilde{\Gamma}^{(4)} = -\lambda + \frac{\lambda^2}{32\pi^2}\left(A_1(s) + A_1(t) + A_1(u) - A_1(4m^2) - 2A_1(0)\right).$$

Higher loop contributions to δ_m, δ_Z and δ_λ are also obtained from the above.

More generally, we can consider the "on shell" renormalization scheme, defined by imposing

$$\Pi'(m^2) = 0, \qquad \frac{d\Pi'}{dp^2}|_{p^2=m^2} = 0, \qquad \tilde{\Gamma}^{(4)}|_{s=\mu} = -\lambda$$

Above we took $\mu = 4m^2$. We could also change the renormalization point μ.

• Now mention two other renormalization schemes, which have an advantage in actual perturbative calculations in that they are *mass independent* (to be illustrated below). In minimal subtraction (MS) we choose the counterterms to remove the $1/\epsilon$ poles, and nothing else. A variant is $\overline{MS}$, where one replaces

$$\lim_{D=4-\epsilon}\frac{\Gamma(2-\frac{1}{2}D)}{(4\pi)^{D/2}(m^2)^{2-\frac{1}{2}D}} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \log(4\pi/m^2)\right)$$

with simply

$$\frac{1}{16\pi^2}\left(\frac{2}{\epsilon} + \log(M^2/m^2)\right),$$

for some arbitrary mass parameter M. (The advantage is that it gets rid of annoying finite constants like γ and other derivatives of the gamma function, which otherwise proliferate at each higher loop order.) The apparent freedom to define things many different ways always cancels out at the end of the day, when one relates to physical observables. Different choices have different benefits along the way.

- Let's consider $\lambda\phi^4$ in MS. To one loop, we have

$$\delta_m = \frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_\lambda = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_Z = 0.$$

Now consider the propagator to two loops. Diagram 1 is a one-loop diagram with the 1-loop $\delta\lambda$ counterterm at the vertex. Diagram 2 is a one-loop diagram with the 1-loop δ_m counterterm on the internal propagator. Diagram 3 is a two-loop diagram which looks like a double-scoop of the 1-loop diagrams. Diagram 4 is a line which cuts through a circle (see your HW). Diagram 5 has no loops, but an insertion of the 2-loop δ_m and δ_Z counter terms. Let's consider the pole terms in the diagrams. Diagram 1 requires no new computation: we can obtain it from the previous 1-loop contribution to $-i\Pi'$ by simply replacing there $\lambda \to \delta\lambda$. This gives

$$-i\Pi'_{diag\ 1} = i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{3}{2}\left(\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{m^2}{4\pi\mu^2} + \frac{1}{\epsilon} - \frac{\gamma}{\epsilon}\right) + \mathcal{O}(\epsilon^0)$$

Diagram 2 has 2 propagators in the loop, with the 1-loop $-i\delta_m$ vertex insertion, which gives (using the integral given at the start, now with $n = 2$ instead of $n = 1$):

$$-i\Pi'_{diag\ 2} = i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{1}{2}\left(\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{m^2}{4\pi\mu^2} - \frac{\gamma}{\epsilon}\right) + \mathcal{O}(\epsilon^0)$$

where the overall $\frac{1}{2}$ is a symmetry factor, as in the 1-loop diagram. Diagram 3 contributes (with two symmetry factors of $\frac{1}{2}$)

$$-i\Pi'_{diag\ 3} = \frac{1}{4}(-i\lambda)^2\mu^{2\epsilon}\int\frac{d^Dk}{(2\pi)^D}\frac{i}{k^2 - m^2}\int\frac{d^Dq}{(2\pi)^D}\left(\frac{i}{q^2 - m^2}\right)^2,$$

where q is the integral over the lower loop, which has two propagators. This gives

$$-i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{1}{2}\left(\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\ln\frac{m^2}{4\pi\mu^2} + \frac{1}{\epsilon} - \frac{2\gamma}{\epsilon}\right) + \mathcal{O}(\epsilon^0)$$

Diagram 4 gives

$$i\frac{\lambda^2}{(16\pi^2)^2}\left(-\frac{m^2}{\epsilon^2} + \frac{1}{\epsilon}\left(m^2\ln\frac{m^2}{4\pi\mu^2} + \frac{1}{12}p^2 + (\gamma - \frac{3}{2}m^2)\right)\right) + \mathcal{O}(\epsilon^0).$$

(The finite (ϵ^0) contribution to diagram 4 can be evaluated by writing out the integrals and using the Feynman trick, but it is quite complicated for general $m \neq 0$. In the HW, you will evaluate it for $m = 0$, where it simplifies.)