2/1/17 Lecture 7 outline

• Last time: Consider isotropic and homogeneous space times: looks the same in all directions and the same under translations. A maximally symmetric space-time of dimension n has

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}),$$

with R the Ricci scalar that is constant over the space-time. The Weyl tensor for these spaces is $C_{\rho\sigma\mu\nu} = 0$. There are three possibilities: $R = 0$: flat; $R > 0$: de Sitter; $R < 0$: anti-de Sitter. These are solutions of Einstein's equations for $T_{\mu\nu} \sim g_{\mu\nu}$, with zero, positive, and negative CC respectively. For $n = 4$ we have $R_{\mu\nu} = 3\kappa g_{\mu\nu}$, where $R = 12\kappa$ and Einstein's equations are satisfied if $\rho = -p = 3\kappa/8\pi G$.

One way to get de Sitter space is to start in 5d, with $ds_5^2 = -du^2+dx^2+dy^2+dz^2+dw^2$ and restrict to a hyperboloid $-u^2 + x^2 + y^2 + z^2 + w^2 = C^2$, where C is the de Sitter radius. By taking $u = C\sinh(t/C)$, and $w, z, y, z \sim C\cosh(t/C)$ times S^3 coordinates, $\sim (\cos\chi, \sin\chi\cos\theta, \sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi)$. The metric is

$$ds^2 = -dt^2 + C^2\cosh^2(t/C)d\Omega_3^2,$$

where $d\Omega_3^2 = d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\phi^2)$ is the solid angle on an S^3. These are geodesically complete coordinates, so the topology is $R \times S^3$.

Defining t' via $\cosh(t/C) = 1/\cos t'$, the metric becomes

$$ds^2 = \frac{C^2}{\cos^2(t')}d\bar{s}^2, \qquad d\bar{s}^2 \equiv -(dt')^2 + d\chi^2 + \sin^2\chi d\Omega_2^2.$$

Here $-\pi/2 < t' < \pi/2$. So de Sitter is conformally related to the metric $d\bar{s}^2$, which is $R \times S^3$; this metric is called the Einstein static universe.

• Conformal, or Carter-Penrose diagrams: find a coordinate change such that, up to a conformal transformation, the original spacetime is related to part of the Einstein static universe. Exhibit the causal and topological structure of the spacetime by drawing a 2d diagram, representing the time and radial coordinate of the conformally-related Einstein static universe subspace. Each point on the diagram is an S^2, except possibly for points on boundary.

The above transformation of de Sitter gives our first example of a Carter-Penrose diagram. We thus represent de Sitter space by a square, with t' on the y axis and $\chi \in [0, \pi]$ on the x axis. Spacelike slices are S^3s, so each point on the diagram is an S^2, except the edges $\chi = 0$ and $\chi = \pi$ are points, the North and South poles of the S^3. Diagonal lines are null rays. So a photon released at past infinity will get to an antipodal point on the sphere at future infinity. Note that points can have disconnected past or future light cones: the spherical spatial sections are expanding so light from one point cannot necessarily get to another.

• Likewise anti-de Sitter starts with $ds_5^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$ and restricts to a hyperboloid $u^2 + v^2 - x^2 - y^2 - z^2 = C^2$. Write $u = C\sin(t')\cosh\rho$, $v = C\cos(t')\cosh\rho$, and $x, y, z \sim C\sinh\rho$ times S^2 coordinates, $\sim (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$, gives

$$ds^2 = C^2(-\cosh^2\rho dt'^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2).$$

The t' coordinate is a closed time-like curve, which is usually undesirable, so instead consider the covering space where t' is not identified with $t' + 2\pi$.

Now do a change of variables $\cosh\rho = 1/\cos\chi$, to find

$$ds^2 = \frac{C^2}{\cos^2\chi} d\bar{s}^2,$$

where $d\bar{s}^2$ is the same Einstein static universe as above. Here $0 \leq \chi < \pi/2$, so anti-de Sitter is conformally related to *half* of the Einstein static universe: the space like slices are topologically a hemisphere of S^3, i.e. R^3. The diagram now looks like a strip, with infinite t' and χ between 0 and $\pi/2$. A point at $\chi = 0$ is at the spatial origin, while one at $\chi = \pi/2$ is an S^2 at spatial infinity. Note that $\chi = \pi/2$, spatial infinity, is time-like, so cannot impose initial value boundary conditions there. A future pointing time-like geodesic can move outward from $t' = 0$, $\chi = 0$ and eventually refocuses to $t' = \pi$, $\chi = 0$.

• Likewise, consider flat Minkowski space-time in spherical coordinates, $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$. It can also be written as conformally equivalent to part of the Einstein static universe. If we picture the Einstein static universe as $R \times S^3$, Minkowski space is a patch where past time-like infinity i^- is $T = -\pi$, $R = 0$; future time-like infinity i^+ is $T = \pi$, $R = 0$, spatial infinity i^0 is $T = 0$, $R = \pi$, future and past null infinity, called scri $\pm$, are $T = \pm(\pi - R)$, for $0 < R < \pi$. The ranges are $0 \leq R < \pi$ and $|T| + R < \pi$. The coordinate change is $T = V + U$, $R = V - U$, $U = \arctan u$, $V = \arctan v$, $u = t - r$, $v = t + r$.