Physics 212b, Ken Intriligator lecture 10, ♡ Feb 14, 2018 ♡

• Time independent, a.k.a. stationary state perturbation theory. Suppose that $H = H_0 + H_1$, with H_0 and H_1 both time-independent. We suppose that $H_1 \ll H_0$, e.g. there is a small parameter ϵ, with $H_0 \sim \epsilon^0$ and $H_1 \sim \epsilon^1$, and that we can do an expansion order-by-order in the small parameter, making corrections to the H_0 case. The scheme only works if the H_0 and H cases are qualitatively similar.

• E.g. a case where this fails is for $H_0 = p^2/2m$ and $H_1 = \epsilon\Theta(a - |x|)$. The energy eigenstates of H_0 are the momentum eigenstates, with arbitrary p, and $E = p^2/2m$. The energy eigenstates for $\epsilon < 0$ include a (parity even) bound state, whereas for $\epsilon > 0$ there is no bound state. The $\epsilon \neq 0$ theory is qualitatively different from that with $\epsilon = 0$. Generally speaking, perturbation theory will break down if there is a qualitative difference between when the small parameter is positive or negative; such a jump across $\epsilon = 0$ indicates that the $\epsilon \to 0$ limit can have subtleties. This also happens in QFT, e.g. in QED if we take the fine structure constant $\alpha = e^2/4\pi\hbar c \approx 1/137$ as the small parameter, perturbation theory works quite well. But $\alpha < 0$ is qualitatively different (non-unitary) showing that perturbation theory can have subtleties (e.g. it is an asymptotic expansion and non-perturbative effects can qualitatively change the results).

• Back to cases where perturbation theory (PT) works. At zero-th order, the energy eigenvalues and eigenstates are $H_0|E_{n,0}\rangle = E_{n,0}|E_{n,0}\rangle$. The full energy eigenstates and eigenvalues are $H|E_n\rangle = E_n|E_n\rangle$. We initially assume that the energy levels are discrete and non-degenerate (more work is required for degenerate perturbation theory).

The unperturbed and perturbed energy eigenstates both form complete orthonormal bases $\mathbf{1} = \sum_n |E_{n,0}\rangle\langle E_{n,0}| = \sum_n |E_n\rangle\langle E_n|$ and $\langle E_n|E_m\rangle = \langle E_{n,0}|E_{m,0}\rangle = \delta_{n,m}$. So $|E_n\rangle = \sum_m |E_{m,0}\rangle\langle E_{m,0}|E_n\rangle$.

Take $H_1 \sim \epsilon$ the small parameter of our perturbation expansion. Then $E_n = \sum_{L=0}^{\infty} E_{n,L}$, where $E_{n,L} \sim \epsilon^L$ and $\langle E_{m,0}|E_n\rangle = \delta_{n,m} + \mathcal{O}(\epsilon)$. Let's write $|E_n\rangle = \sum_{L=0}^{\infty} |E_{n,L}\rangle$ where we think of $|E_{n,L}\rangle \sim \epsilon^L$.

• Expand $(H_0 + H_1)|E_n\rangle = E_n|E_n\rangle$ in ϵ. To order ϵ^1 we have $H_0|E_{n,1}\rangle + H_1|E_{n,0}\rangle = E_{n,0}|E_{n,1}\rangle + E_{n,1}|E_{n,0}\rangle$. Therefore $E_{n,1} = \langle E_{n,0}|H_1|E_{n,0}\rangle$ via projecting on to $\langle E_{m\neq n,0}|$:

$$\langle E_{m,0}|H_1|E_{n,0}\rangle = (E_{n,0} - E_{m,0})\langle E_{m,0}|E_{n,1}\rangle \qquad (m \neq n).$$

The normalization condition to order ϵ implies that $\langle E_{n,1}|E_{n,0}\rangle = iC\epsilon$ where C is a real, order 1 constant. To this order, $1 + iC\epsilon = e^{iC\epsilon}$ and we can eliminate this as an overall phase of $|E_n\rangle$, so the first order correction can be taken to be purely orthogonal to the zero-th order term:

$$|E_{n,1}\rangle = \sum_{m\neq n} \frac{|E_{m,0}\rangle\langle E_{m,0}|H_1|E_{n,0}\rangle}{E_{n,0} - E_{m,0}} = \frac{P_{n\perp}}{E_{n,0} - H_0} H_1 |E_{n,0}\rangle$$

where $P_{n\perp} \equiv 1 - |E_{n,0}\rangle\langle E_{n,0}|$.

- At $\mathcal{O}(\epsilon^2)$: $H_0|E_{n,2}\rangle + H_1|E_{n,1}\rangle = E_{n,0}|E_{n,2}\rangle + E_{n,1}|E_{n,1}\rangle + E_{n,2}|E_{n,0}\rangle$. Multiply both sides by $\langle E_{n,0}|$ to get $\langle E_{n,0}|H_1|E_{n,1}\rangle = E_{n,2}$ so

$$E_{n,2} = \sum_{m\neq n} \frac{|\langle E_{m,0}|H_1|E_{n,0}\rangle|^2}{E_{n,0} - E_{m,0}}.$$

Note that this is always negative for the ground state.